HL

Now with interactive e-book

PRACTICE QUESTIONS

PEARSON BACCALAUREATE

HIGHER LEVEL

Mathematics

2012 edition

DEVELOPED SPECIFICALLY FOR THE

IB DIPLOMA

IBRAHIM WAZIR • TIM GARRY

PETER ASHBOURNE • PAUL BARCLAY • PETER FLYNN • KEVIN FREDERICK • MIKE WAKEFORD

ALWAYS LEARNING

PEARSON

Chapter 2

Practice questions

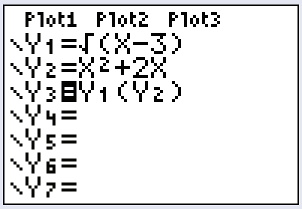
1 $f(g(x)) = f\left(x^2 + 2x\right) = \sqrt{x^2 + 2x - 3}$

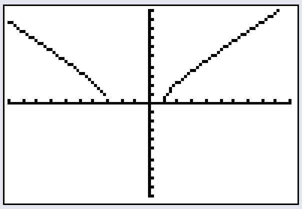
a A radical function is not defined when the expression under the square root has a negative value, so:

$x^2 + 2x - 3 < 0 \Rightarrow x^2 - x + 3x - 3 < 0 \Rightarrow x(x-1) + 3(x-1) < 0 \Rightarrow (x-1)(x+3) < 0 \Rightarrow x \in]-3, 1[$

Therefore, $a = -3$, $b = 1$.

b





The range of the function is $y \geqslant 0$.

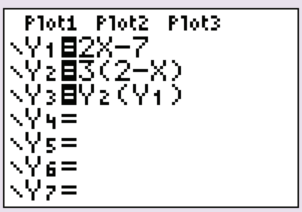
Solution Paper 1 type

2 **a** $g^{-1}(3) = a \Rightarrow g(a) = 3 \Rightarrow 2a - 7 = 3 \Rightarrow 2a = 3 + 7 \Rightarrow a = 5 \Rightarrow g^{-1}(3) = 5$

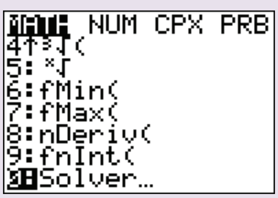
b $h(g(6)) = h(2 \times 6 - 7) = h(5) = 3(2-5) = 3 \times -3 = -9$

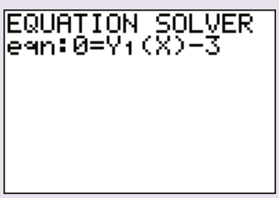
Solution Paper 2 type

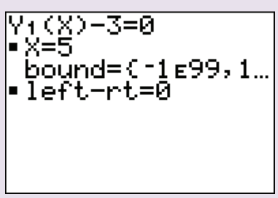
2 Firstly, we are going to input both functions in the Function menu.



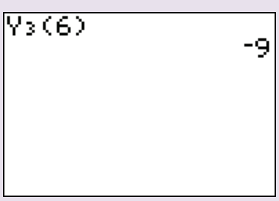
a We have to solve the equation $g(x) = 3$, which can be solved in many different ways. Here we are going to use Solver.







b We are going to define the third function as a composition, $y = g(h(x))$, and calculate its value for $x = 6$.



Solution Paper 1 type

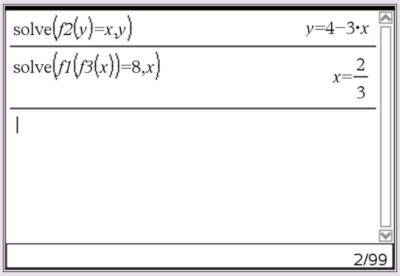
3 **a** $g(x) = \dfrac{4-x}{3} \Rightarrow x = \dfrac{4-y}{3} \Rightarrow 3x = 4 - y \Rightarrow y = 4 - 3x \Rightarrow g^{-1}(x) = 4 - 3x$

b $f\left(g^{-1}(x)\right) = f(4-3x) = 5(4-3x) - 2 = 20 - 15x - 2 = 18 - 15x \Rightarrow 18 - 15x = 8 \Rightarrow 10 = 15x \Rightarrow x = \dfrac{\cancel{10}\,2}{\cancel{15}\,3} = \dfrac{2}{3}$

Solution Paper 2 type

3 This question can be solved using a CAS calculator, which, at present, is not permitted on the final exam by the IBO.

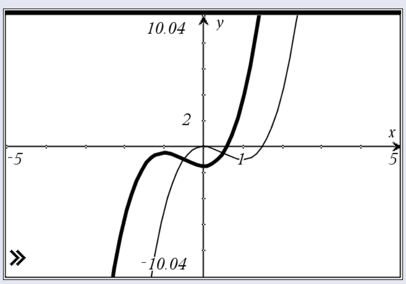
a–b



4 **a** $g(h(x)) = g(2x) = 2x - 3$

b $\left.\begin{aligned} g(y) = x \Rightarrow y - 3 = x \Rightarrow g^{-1}(x) = x + 3 \\ h(y) = x \Rightarrow 2y = x \Rightarrow h^{-1}(x) = \frac{x}{2} \end{aligned}\right\} \Rightarrow g^{-1}(14) + h^{-1}(14) = 17 + 7 = 24$

5 **a**



b Since the function is transformed by applying a horizontal translation of one unit to the left and a vertical translation of half a unit down, the new coordinates of the maximum point are $\left(-1, -\dfrac{1}{2}\right)$, whilst the new coordinates of the minimum point are $\left(0, -\dfrac{3}{2}\right)$.

6 There are four transformations defined by the graph $y = -\dfrac{1}{2}(x+5)^2 + 3$. The first is a reflection in the x-axis (the line $y = 0$). The second is a vertical stretch by scale factor $\dfrac{1}{2}$. Then we have two translations: a horizontal translation of five units to the left and a vertical translation of three units up. Therefore, the parameters are as follows:

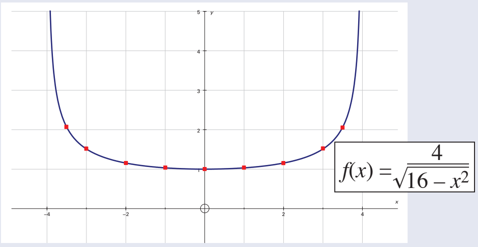
a $k = \dfrac{1}{2}$ **b** $p = -5$ **c** $q = 3$

7 Since we cannot use a calculator or apply any calculus knowledge, as this appears later in the textbook, we will simply calculate the corresponding y-values for the given x-values from the domain. We also notice that this is an even function, which means $f(-x) = \frac{4}{\sqrt{16-(-x)^2}} = \frac{4}{\sqrt{16-x^2}} = f(x)$, and geometrically its graph is symmetrical with respect to the y-axis.

a Using a scientific calculator we obtain the following y-values:

x	–3.5	–3	–2	–1	0	1	2	3	3. 5
y	2.07	1.51	1.15	1.03	1	1.03	1.15	1.51	2.07

From the table, we see that $f(x)$ has a minimum value when the denominator is a maximum, i.e. when $x = 0$.



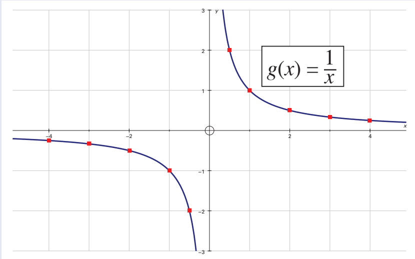
b Since $16 - x^2 > 0$, the vertical asymptotes are $x = -4$ and $x = 4$.

c Looking at the table and the graph, we can see that the lowest possible y-value is 1 and the graph has two vertical asymptotes. Therefore, when the values of x approach –4 or 4, the y-values tend to positive infinity.

$R(f) = \{y \in \mathbb{R} : y \geqslant 1\}$

8 **a** We are going to use a tabular form to show some points on the graph.

x	–4	–3	–2	–1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	–1	–2	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



b The horizontal translation of four units to the left means that instead of x we will have the expression $(x + 4)$; whilst the vertical translation of two units down means that we are going to subtract 2 at the end. So, the function h looks like this: $h(x) = \frac{1}{x+4} - 2$.

Note: This function can be written in a different form: $h(x) = \frac{1}{x+4} - 2 = \frac{1-2x-8}{x+4} = -\frac{2x+7}{x+4}$.

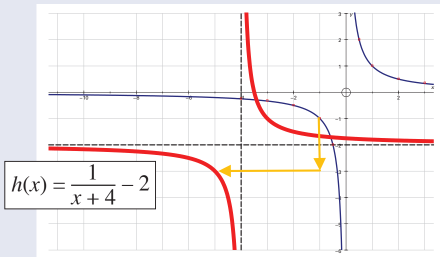
This second form might be better for the calculations in part **c**.

c **i** $y = 0 \Rightarrow -\frac{2x+7}{x+4} = 0 \Rightarrow 2x+7 = 0 \Rightarrow x = -\frac{7}{2} \Rightarrow \left(-\frac{7}{2}, 0\right)$

$x = 0 \Rightarrow y = -\frac{2\times 0+7}{0+4} = -\frac{7}{4} \Rightarrow \left(0, -\frac{7}{4}\right)$

ii By looking at the first form, we see that the vertical asymptote has the equation $x = -4$, whilst the horizontal asymptote has the equation $y = -2$. That result can also be found by looking at the asymptotes of the original function, $y = \frac{1}{x}$, which were the coordinate axes. By translating the original function four units to the left, we were also translating the vertical asymptote. And again, by translating the function two units down, we were translating the horizontal asymptote.

iii



9 a i $f(8) = \sqrt{8+3} = \sqrt{11}$

ii $f(46) = \sqrt{46+3} = \sqrt{49} = 7$

iii $f(-3) = \sqrt{-3+3} = \sqrt{0} = 0$

b The function f is not defined for negative values under the square root, and therefore $x + 3 < 0 \Rightarrow x < -3$.

c $g(f(x)) = g\left(\sqrt{x+3}\right) = \left(\sqrt{x+3}\right)^2 - 5 = x + 3 - 5 = x - 2, x \geqslant -3$

Notice here that the composition would not be represented by the whole straight line, but simply by a ray starting at the point $(-3, -5)$.

10 a $g^{-1}(-2) = a \Rightarrow g(a) = -2 \Rightarrow \frac{a-8}{2} = -2 \Rightarrow a - 8 = -4 \Rightarrow a = 4$

An alternative method would be to find the inverse function first and then calculate its value at -2.

$g(y) = x \Rightarrow \frac{y-8}{2} = x \Rightarrow y - 8 = 2x \Rightarrow g^{-1}(x) = 2x + 8$

(This calculation is going to be used in the following part.)

$g^{-1}(-2) = 2 \times (-2) + 8 = 4$

b $g^{-1}(h(x)) = g^{-1}\left(x^2 - 1\right) = 2 \times \left(x^2 - 1\right) + 8 = 2x^2 - 2 + 8 = 2x^2 + 6$

c $g^{-1}(h(x)) = 22 \Rightarrow 2x^2 + 6 = 22 \Rightarrow 2x^2 = 16 \Rightarrow x^2 = 8 \Rightarrow x = \pm\sqrt{8} = \pm 2\sqrt{2}$

11 a $f(y) = x \Rightarrow 3y - 1 = x \Rightarrow 3y = x + 1 \Rightarrow y = \frac{x+1}{3}$

b $f(g(x)) = f\left(\frac{4}{x}\right) = 3 \times \left(\frac{4}{x}\right) - 1 = \frac{12}{x} - 1$

c $f(g(y)) = x \Rightarrow \frac{12}{y} - 1 = x \Rightarrow \frac{12}{y} = x + 1 \Rightarrow (f \circ g(x))^{-1} = \frac{12}{x+1}, x \neq -1$

d $g(g(x)) = g\left(\frac{4}{x}\right) = \frac{4}{\frac{4}{x}} = \cancel{4} \times \frac{x}{\cancel{4}} = x$

We notice that the function is self-inverse. That can be spotted from the graph, which is symmetrical with respect to the line $y = x$.

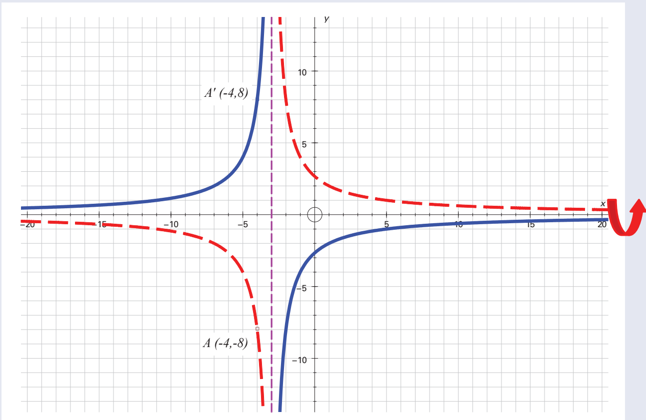
12 a Even though the question suggests finding the parameters in a different order, we are going to use the form of the function first and then the point.

ii Since the vertical line MN is a vertical asymptote, we can read its equation as $x = -3$. The function of the form $h(x) = \frac{a}{x-b}$ has a vertical asymptote when the denominator is equal to zero, so we can find the value of b: $b = -3$.

i To find parameter a, we will use point $A(-4, -8)$ which is on the graph.

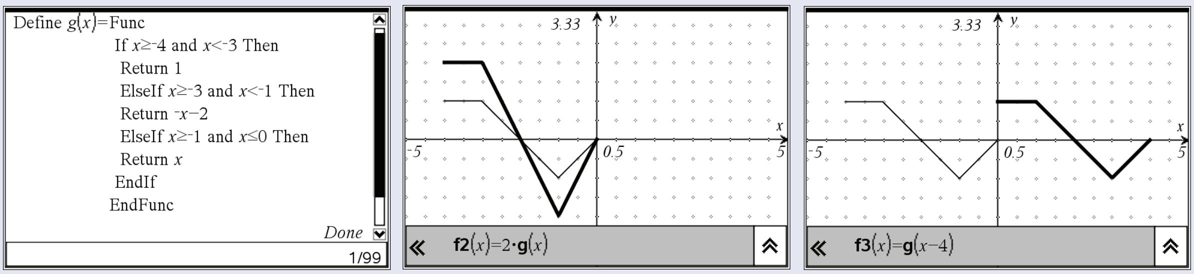
$$-8 = \frac{a}{-4+3} \Rightarrow a = -8 \times (-1) = 8$$

b

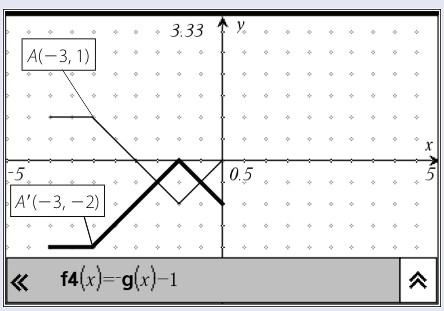


The transformation is a reflection in the x-axis.

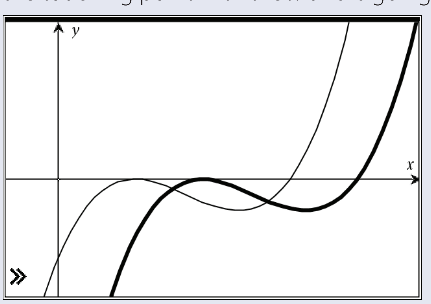
13 a



b To obtain the graph $y = -f(x) - 1$ we need to perform two transformations: a reflection in the x-axis and then a vertical translation of one unit down. Therefore, point $A(-3, 1)$ will first be reflected to $(-3, -1)$ and then translated vertically to $A'(-3, -2)$.



14 The graph of the function $y_2 = f(x - k)$ is obtained by a horizontal translation of k units to the right. Since $0 < k < n - m$, the touching point with the x-axis is going to be translated horizontally and placed before the zero n.



15 Firstly, we need to find the composite function and then we will find its inverse.

$(f \circ g)(x) = f(g(x)) = f(x^3) = x^3 + 1 \Rightarrow x = y^3 + 1 \Rightarrow x - 1 = y^3 \Rightarrow (f \circ g)^{-1}(x) = \sqrt[3]{x-1}$

16 a $g(x) = (f \circ f)(x) = f(f(x)) = f\left(\dfrac{x}{x+1}\right) = \dfrac{\dfrac{x}{x+1}}{\dfrac{x}{x+1}+1} = \dfrac{\dfrac{x}{\cancel{x+1}}}{\dfrac{x+x+1}{\cancel{x+1}}} = \dfrac{x}{2x+1},\ x \neq -1, -\dfrac{1}{2}$

b $(g \circ g)(2) = g(g(2)) = g\left(\dfrac{2}{5}\right) = \dfrac{\dfrac{2}{5}}{2 \times \dfrac{2}{5} + 1} = \dfrac{2}{9}$

17 a $f(x) = \sqrt{\dfrac{1}{x^2} - 2} \Rightarrow \dfrac{1}{x^2} - 2 \geqslant 0 \Rightarrow \dfrac{1-2x^2}{x^2} \geqslant 0 \Rightarrow 1 - 2x^2 \geqslant 0 \Rightarrow -\dfrac{\sqrt{2}}{2} \leqslant x \leqslant \dfrac{\sqrt{2}}{2},\ x \neq 0$

b Regarding the range, we notice that the sign before the square root is positive, and therefore $f(x) \geqslant 0$.

18 $x = \dfrac{2y+1}{y-1} \Rightarrow xy - x = 2y + 1 \Rightarrow xy - 2y = x + 1 \Rightarrow y(x-2) = x + 1 \Rightarrow f^{-1}(x) = \dfrac{x+1}{x-2}$

Since the denominator cannot be equal to zero, the domain of the inverse function is $D(f^{-1}) = \{x \in \mathbb{R} \mid x \neq 2\}$.

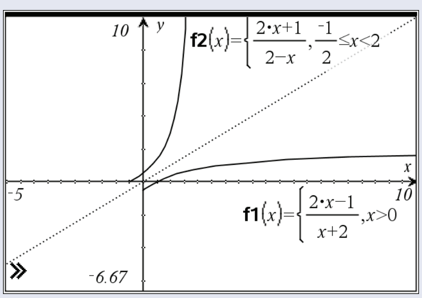
19 a The rational function $f(x) = \dfrac{2x-1}{x+2}$, $x > 0$, has a horizontal asymptote at $y = 2$ because

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{2x-1}{x+2} = \lim\limits_{x\to\infty} \dfrac{2 - \overbrace{\frac{1}{x}}^{0}}{1 + \underbrace{\frac{2}{x}}_{0}} = \dfrac{2}{1} = 2$. On the other hand, since the restriction on the domain is to positive values of x, and the vertical asymptote without restriction would occur at $x = -2$, the minimum on the function takes place at $x = 0 \Rightarrow y = -\dfrac{1}{2}$.

Therefore, the range is $R(f) = \left\{y \in \mathbb{R} \,\middle|\, -\dfrac{1}{2} \leqslant y < 2\right\}$.

b $x = \dfrac{2y-1}{y+2} \Rightarrow xy + 2x = 2y - 1 \Rightarrow 2x + 1 = 2y - xy \Rightarrow f^{-1}(x) = \dfrac{2x+1}{2-x}$. Now, since there was a restriction on the domain and range of the original function, we have a restriction on the domain of the inverse function:

$D(f^{-1}) = \left\{x \in \mathbb{R} \,\middle|\, -\dfrac{1}{2} \leqslant x < 2\right\}$.



20 a If $f(x) = x^3$, then $f(g(x)) = x + 1 \Rightarrow (g(x))^3 = x + 1 \Rightarrow g(x) = \sqrt[3]{x+1}$.

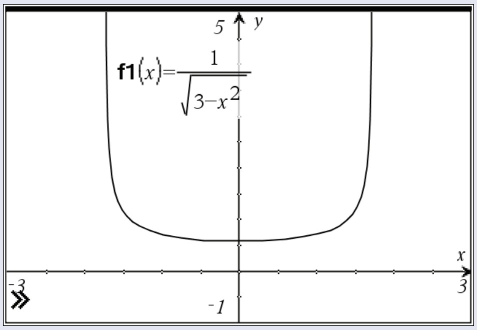
b $g(f(x)) = x + 1 \Rightarrow g(x^3) = x + 1 \Rightarrow g(x) = \sqrt[3]{x} + 1$

21 a Since the surd expression is in the denominator, it cannot be equal to zero and hence:

$3 - x^2 > 0 \Rightarrow D(f) = \{x \in \mathbb{R} \mid -\sqrt{3} < x < \sqrt{3}\}$

b The maximum value of the expression in the denominator is $\sqrt{3}$. Therefore, the range of the function is

$$R(f) = \left\{ y \in \mathbb{R} \,\middle|\, x \geqslant \frac{\sqrt{3}}{3} \right\}.$$



22 Given that $(f \circ g)(x) = \frac{x+1}{2}$ and $g(x) = 2x - 1$, we will firstly find $f(x)$:

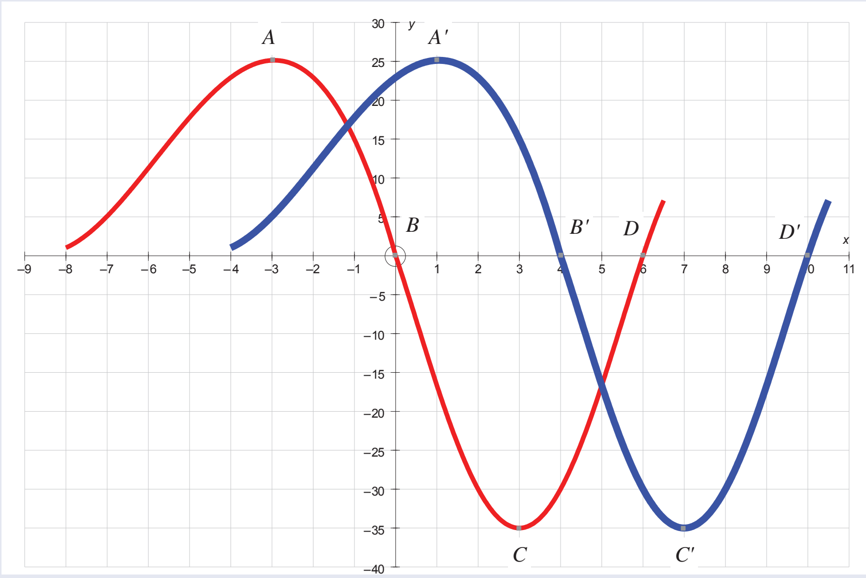
$$f(g(x)) = \frac{x+1}{2} \quad \Rightarrow f(2x-1) = \frac{x+1}{2}$$

To solve this type of functional equation, we will use a substitution, $2x - 1 = t \Rightarrow x = \frac{t+1}{2}$, and then transform the rule to the new variable.

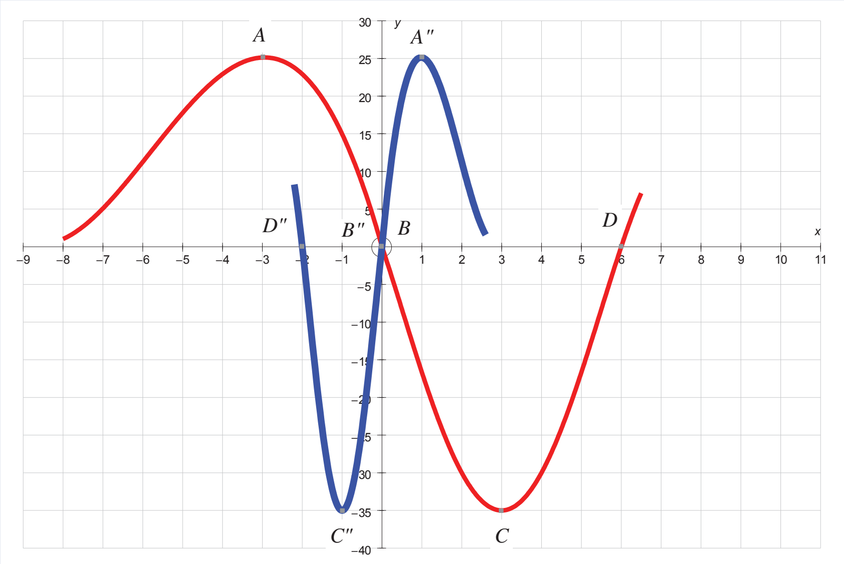
$$f(t) = \frac{\frac{t+1}{2}+1}{2} = \frac{\frac{t+1+2}{2}}{2} = \frac{t+3}{4} \Rightarrow f(x-3) = \frac{x-3+3}{4} = \frac{x}{4}$$

23 a The first transformation, $y = f(x-4)$, is a horizontal translation of four units to the right. Therefore,

$A(-3, 25) \to A'(1, 25)$; $B(0, 0) \to B'(4, 0)$; $C(3, -35) \to C'(7, -35)$; and $D(6, 0) \to D'(10, 0)$.



b The second transformation, $y = f(-3x)$, actually consists of two transformations. One transformation is a reflection in the y-axis, whilst the other is a horizontal shrinking by scale factor 3. So, the points will be mapped as follows: $A(-3, 25) \to A''(1, 25)$; $B(0, 0) \to B''(0, 0)$; $C(3, -35) \to C''(-1, -35)$; and $D(6, 0) \to D''(-2, 0)$.



Chapter 3

Practice questions

1 $x^2 - (a+3b)x + 3ab = 0$

$$x_{1,2} = \frac{(a+3b) \pm \sqrt{(a+3b)^2 - 4\cdot 3ab}}{2} = \frac{(a+3b) \pm \sqrt{a^2 + 6ab + 9b^2 - 12ab}}{2}$$

$$= \frac{(a+3b) \pm \sqrt{a^2 - 6ab + 9b^2}}{2} = \frac{(a+3b) \pm \sqrt{(a-3b)^2}}{2} = \frac{(a+3b) \pm (a-3b)}{2}$$

$$x_1 = \frac{a+3b+a-3b}{2} = a,\ x_2 = \frac{a+3b-a+3b}{2} = 3b$$

2 $\dfrac{3x-2}{5} + 3 \geqslant \dfrac{4x-1}{3} \Big/ \times 15$

$9x - 6 + 45 \geqslant 20x - 5$

$-11x \geqslant -44$

$x \leqslant 4$

3 If, for parabola $y = f(x) = 3x^2 - 8x + c$, the vertex is at $\left(\frac{4}{3}, -\frac{1}{3}\right)$, then:

$$f\left(\frac{4}{3}\right) = -\frac{1}{3} \Rightarrow 3\cdot\left(\frac{4}{3}\right)^2 - 8\cdot\left(\frac{4}{3}\right) + c = -\frac{1}{3} \Rightarrow \frac{16}{3} - \frac{32}{3} + c = -\frac{1}{3} \Rightarrow c = \frac{15}{3} = 5$$

4 The quadratic function $f(x) = ax^2 + bx + c$:

- **i** passes through $(2, 4) \Rightarrow f(2) = 4$
- **ii** has a maximum value of 6 when $x = 4 \Rightarrow f(4) = 6$
- **iii** has a zero of $x = 4 + 2\sqrt{3} \Rightarrow f(4 + 2\sqrt{3}) = 0$

The zeros of the quadratic function are symmetrical about the axis of symmetry, i.e. $x = 4$, so the other zero is $x = 4 - 2\sqrt{3}$. The function can be written as:

$$f(x) = a\left(x - \left(4 + 2\sqrt{3}\right)\right)\left(x - \left(4 - 2\sqrt{3}\right)\right) = a\left(x - 4 - 2\sqrt{3}\right)\left(x - 4 + 2\sqrt{3}\right)$$

$$= a\left((x-4)^2 - \left(2\sqrt{3}\right)^2\right) = a\left(x^2 - 8x + 16 - 12\right) = a\left(x^2 - 8x + 4\right)$$

Since $f(2) = 4 \Rightarrow a(4 - 16 + 4) = 4 \Rightarrow -8a = 4 \Rightarrow a = -\frac{1}{2}$

The function is: $f(x) = -\frac{1}{2}\left(x^2 - 8x + 4\right) = -\frac{1}{2}x^2 + 4x - 2 \Rightarrow a = -\frac{1}{2}, b = 4, c = -2.$

5 The equation $x^3 + 5x^2 + px + q = 0$ can be written in the factorized form as:

$$(x-\omega)(x-2\omega)(x-\omega-3) = 0 \Rightarrow \left(x^2 - 3\omega x + 2\omega^2\right)(x - \omega - 3) = 0 \Rightarrow$$

$$x^3 - 3\omega x^2 + 2\omega^2 x - \omega x^2 + 3\omega^2 x - 2\omega^3 - 3x^2 + 9\omega x - 6\omega^2 = 0$$

$$x^3 + (-4\omega - 3)x^2 + \left(5\omega^2 + 9\omega\right)x + \left(-2\omega^3 - 6\omega^2\right) = 0$$

By comparing the corresponding coefficients, we obtain the following system of equations:

$$\begin{cases} -4\omega - 3 = 5 \\ 5\omega^2 + 9\omega = p \\ -2\omega^3 - 6\omega^2 = q \end{cases} \Rightarrow \begin{cases} \omega = -2 \\ p = 2 \\ q = -8 \end{cases}$$

6 The discriminant of the equation $mx^2 - 2(m+2)x + m + 2 = 0$ is

$\Delta = (-2(m+2))^2 - 4m(m+2) = 4(m^2 + 4m + 4) - 4m^2 - 8m = 8m + 16.$

a The equation has two real roots when $\Delta > 0 \;\Rightarrow 8m + 16 > 0 \;\Rightarrow m > -2.$

b The equation has one positive and one negative real root when $\Delta > 0$ and the first and the last coefficients are of opposite sign. We have two possibilities:

$m > 0$ and $m + 2 < 0 \Rightarrow m > 0$ and $m < -2$, which is not possible; or

$m < 0$ and $m + 2 > 0 \Rightarrow m < 0$ and $m < -2$, which gives us the solution $-2 < m < 0$.

7 For the polynomial $f(x) = x^3 + ax^2 + bx + c$, we are told that:

i $x - 1$ is a factor $\Rightarrow f(1) = 0$

ii $x + 1$ is a factor $\Rightarrow f(-1) = 0$

iii division by $x - 2$ gives a remainder of $12 \Rightarrow f(2) = 12$

Therefore, we have the following system of equations:

$$\begin{cases} 1 + a + b + c = 0 \\ -1 + a - b + c = 0 \\ 8 + 4a + 2b + c = 0 \end{cases} \Rightarrow \begin{cases} a + b + c = -1 \\ a - b + c = 1 \\ 4a + 2b + c = -8 \end{cases} \Rightarrow \begin{cases} -2b = 2 \\ 3a + b = -7 \end{cases} \Rightarrow \begin{cases} b = -1 \\ 3a - 1 = -7 \Rightarrow a = -2 \end{cases} \Rightarrow -8 - 2 + c = -8 \Rightarrow c = 2$$

So: $a = -2, b = -1, c = 2.$

8 $|x| < 5\,|x - 6|$

Since the expressions on both sides must be positive, we can square both sides and remove the absolute value signs.

$x^2 < 25\,(x - 6)^2$

$x^2 < 25x^2 - 300x + 900$

$24x^2 - 300x + 900 > 0$

$6x^2 - 75x + 225 > 0$

$$x_{1,2} = \frac{75 \pm \sqrt{(-75)^2 - 4 \cdot 6 \cdot 225}}{2 \cdot 6} = \frac{75 \pm 15}{12}$$

$$x_1 = \frac{90}{12} = \frac{15}{2},\ x_2 = \frac{60}{12} = 5$$

$$(x - 5)\left(x - \frac{15}{2}\right) > 0$$

We analyze the signs of both factors in a 'sign chart':

		5		$\frac{15}{2}$		x
$x - 5$	$-$	$\mathbf{0}$	$+$		$+$	
$x - \frac{15}{2}$	$-$		$-$	$\mathbf{0}$	$+$	
$(x-5)\left(x - \frac{15}{2}\right)$	$+$	$\mathbf{0}$	$-$	$\mathbf{0}$	$+$	

The solution is: $x < 5$ or $x > \frac{15}{2}$.

9 The equation $2x^2 + (3-k)x + k + 3 = 0$ has two imaginary solutions when its discriminant is negative.

$\Delta = (3-k)^2 - 4 \cdot 2(k+3) = 9 - 6k + k^2 - 8k - 24 = k^2 - 14k - 15$

$k^2 - 14k - 15 < 0$

$(k+1)(k-15) < 0$

We analyze the signs of both factors in a 'sign chart':

x		-1		15	
$k+1$	$-$	$\mathbf{0}$	$+$		$+$
$k-15$	$-$		$-$	$\mathbf{0}$	$+$
$(k+1)(k-15)$	$+$	$\mathbf{0}$	$-$	$\mathbf{0}$	$+$

The solution is: $-1 < k < 15$.

10 a $f(x) = \dfrac{2x^2 + 8x + 7}{x^2 + 4x + 5} = \dfrac{2x^2 + 8x + 10 - 3}{x^2 + 4x + 5} = \dfrac{2\left(x^2 + 4x + 5\right) - 3}{x^2 + 4x + 5}$

$= 2 - \dfrac{3}{x^2 + 4x + 5} = 2 - \dfrac{3}{\left(x^2 + 4x + 4\right) + 1} = 2 - \dfrac{3}{(x+2)^2 + 1}$

b i $\lim\limits_{x \to +\infty} f(x) = 2$

ii $\lim\limits_{x \to -\infty} f(x) = 2$

c The minimum value occurs when $\dfrac{3}{(x+2)^2 + 1}$ is largest, that is, when $x = -2$.

$f(-2) = 2 - \dfrac{3}{(-2+2)^2 + 1} = -1$

The minimum point is $(-2, -1)$.

11 The equation $(k-2)x^2 + 4x - 2k + 1 = 0$ has two distinct real roots when its discriminant is positive.

$\Delta = 4^2 - 4(k-2)(-2k+1) = 16 - 4\left(-2k^2 - 4k - 2\right) = 8k^2 + 16k + 24$

$8k^2 + 16k + 24 > 0$

$k^2 + 2k + 3 > 0$

In order to factorize this inequality, we try to solve the equation $k^2 + 2k + 3 = 0$.

$k_{1,2} = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2} = \dfrac{-2 \pm \sqrt{-8}}{2}$

Since this equation has no real solutions, the inequality is valid for all $k \in \mathbb{R}$; so, the original equation has two distinct real roots for all $k \in \mathbb{R}$.

12 When $f(x) = 6x^4 + 11x^3 - 22x^2 + ax + 6$ is divided by $(x+1)$ the remainder is -20.

$f(-1) = -20 \Rightarrow 6 - 11 - 22 - a + 6 = -20 \Rightarrow a = -1$

13 When $p(x) = (ax + b)^3$ is divided by $(x + 1)$ the remainder is $-1 \Rightarrow p(-1) = -1$.

When divided by $(x - 2)$, the remainder is $27 \Rightarrow p(2) = 27$.

Therefore, we have the following system of equations:

$$\begin{cases}(-a+b)^3 = -1 \\ (2a+b)^3 = 27\end{cases} \Rightarrow \begin{cases}-a+b=-1 \\ 2a+b=3\end{cases} \Rightarrow 3a = 4 \Rightarrow a = \frac{4}{3}, b = \frac{1}{3}$$

14 When $f(x) = x^3 + 3x^2 + ax + b$ is divided by $(x + 1)$ the remainder is the same as that when divided by $(x - 2)$.

$f(-1) = f(2)$

$-1 + 3 - a + \cancel{b} = 8 + 12 + 2a + \cancel{b}$

$2 - a = 20 + 2a$

$3a = -18$

$a = -6$

15 When $f(x) = x^4 + ax + 3$ is divided by $(x - 1)$ the remainder is 8.

$f(1) = 8 \Rightarrow 1 + a + 3 = 8 \Rightarrow a = 4$

16 The polynomial $g(x) = x^3 + ax^2 - 3x + b$ is divisible by $(x - 2) \Rightarrow g(2) = 0$.

When divided by $(x + 1)$ the remainder is $6 \Rightarrow g(-1) = 6$.

We have the following system of equations:

$$\begin{cases}8 + 4a - 6 + b = 0 \\ -1 + a + 3 + b = 6\end{cases} \Rightarrow \begin{cases}4a + b = -2 \\ a + b = 4\end{cases} \Rightarrow 3a = -6 \Rightarrow a = -2, b = 6$$

17 The polynomial $g(x) = x^2 - 4x + 3$ can be factorized as $(x-1)(x-3)$, so if $g(x)$ is a factor of $f(x) = x^3 + (a-4)x^2 + (3-4a)x + 3$, then both $(x-1)$ and $(x-3)$ are factors of $f(x)$. We can use any of them in further calculations:

$f(1) = 0 \Rightarrow 1 + a - 4 + 3 - 4a + 3 = 0 \Rightarrow -3a = -3 \Rightarrow a = 1$

18 Given that $(x + 2)$ is a factor of $f(x) = x^3 - 2x^2 - 5x + k$:

$f(-2) = 0 \Rightarrow -8 - 8 + 10 + k = 0 \Rightarrow k = 6$

19 If $1 + ki$ is one zero of the polynomial $z^2 + kz + 5$, then $1 - ki$ is the second, and the polynomial can be written as:

$(z - (1+ki))(z - (1-ki)) = ((z-1) - ki)((z-1) + ki) = (z-1)^2 - (ki)^2 = z^2 - 2z + 1 + k^2$

By comparing the coefficients, we see that $k = -2$ and $k^2 + 1 = 5$.

20 The equation $kx^2 - 3x + (k + 2) = 0$ has two distinct real roots when its discriminant is positive.

$\Delta = (-3)^2 - 4k(k+2) = 9 - 4k^2 - 8k$

$9 - 4k^2 - 8k > 0$

$4k^2 + 8k - 9 < 0$

$$k_{1,2} = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 4 \cdot (-9)}}{2 \cdot 4} = \frac{-8 \pm \sqrt{208}}{8} = \frac{-8 \pm 4\sqrt{13}}{8} = \frac{-2 \pm \sqrt{13}}{4}$$

$k_1 \approx -2.80, k_2 \approx 0.803$

$(k + 2.8)(k - 0.803) < 0$

We analyze the signs of both factors in a 'sign chart':

x		-2.80		0.803	
$k+2.80$	$-$	0	$+$		$+$
$k-0.803$	$-$		$-$	0	$+$
$(k+2.80)(k-0.803)$	$+$	0	$-$	0	$+$

The solution is: $-2.80 < k < 0.803$.

21 The equation $(1+2k)x^2 - 10x + k - 2 = 0$ has real roots when its discriminant is not negative.

$\Delta = (-10)^2 - 4(1+2k)(k-2) = 100 - 4(2k^2 - 3k - 2) = -8k^2 + 12k + 108$

$-8k^2 + 12k + 108 \geqslant 0$

$2k^2 - 3k - 27 \leqslant 0$

$$k_{1,2} = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot (-27)}}{2 \cdot 2} = \frac{3 \pm 15}{4}$$

$k_1 = -3, k_2 = \dfrac{9}{2}$

$2(k+3)\left(k - \dfrac{9}{2}\right) \leqslant 0$

$(k+3)\left(k - \dfrac{9}{2}\right) \leqslant 0$

We analyze the signs of both factors in a 'sign chart':

x		-3		$\frac{9}{2}$	
$k+3$	$-$	0	$+$		$+$
$k-\frac{9}{2}$	$-$		$-$	0	$+$
$(k+3)\left(k-\frac{9}{2}\right)$	$+$	0	$-$	0	$+$

The solution is: $-3 \leqslant k \leqslant \dfrac{9}{2}$.

22 To determine the range of values of m such that, for all real x, $m(1+x) \leqslant x^2$ we transform the inequality into $x^2 - mx - m \geqslant 0$ and observe the function $f(x) = x^2 - mx - m$. The problem is now $f(x) \geqslant 0$. Since the leading coefficient is 1, the graph of this quadratic function opens upwards; thus:

$\Delta \leqslant 0 \Rightarrow (-m)^2 - 4 \cdot 1 \cdot (-m) \leqslant 0 \Rightarrow m^2 + 4m \leqslant 0 \Rightarrow m(m+4) \leqslant 0$

We analyze the signs of both factors in a 'sign chart':

		-4		0		x
$m+4$	$-$	0	$+$		$+$	
m	$-$		$-$	0	$+$	
$m(m+4)$	$+$	0	$-$	0	$+$	

The solution is: $-4 \leqslant m \leqslant 0$.

23 $|5-3x| \leqslant |x+1|$

Since the expressions on both sides must be positive, we can square both sides and remove the absolute value signs.

$(5-3x)^2 \leqslant (x+1)^2$

$25-30x+9x^2 \leqslant x^2+2x+1$

$8x^2-32x+24 \leqslant 0$

$x^2-4x+3 \leqslant 0$

$(x-1)(x-3) \leqslant 0$

We analyze the signs of both factors in a 'sign chart':

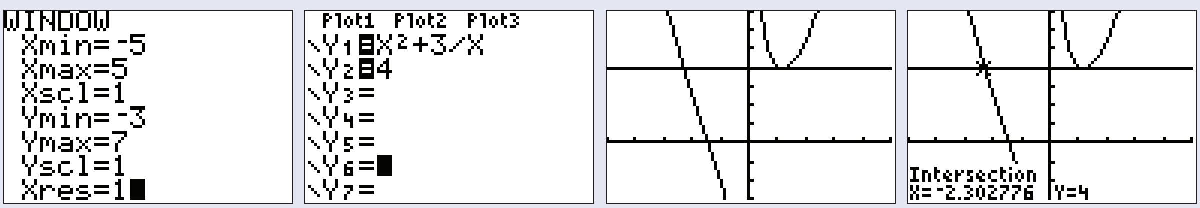
		1		3		x
$x-1$	$-$	0	$+$		$+$	
$x-3$	$-$		$-$	0	$+$	
$(x-1)(x-3)$	$+$	0	$-$	0	$+$	

The solution is: $1 \leqslant x \leqslant 3$.

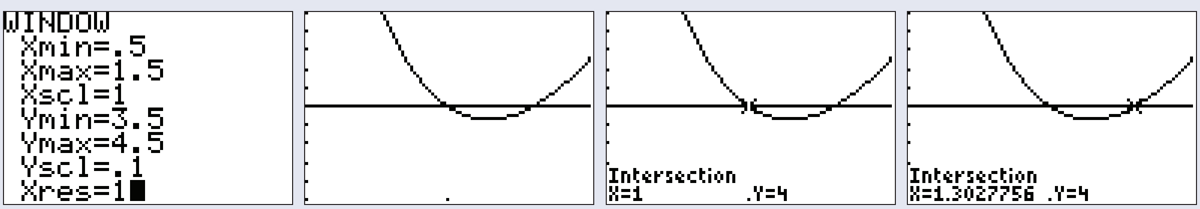
24 $x^2-4+\dfrac{3}{x}<0$

$x^2+\dfrac{3}{x}<4$

We will look for $x \in \mathbb{R}$ so that the graph of the function $f(x)=x^2+\dfrac{3}{x}$ is below the line $y=4$.



For the right branch, we must change the window to see the intersections.



The solution is: $-2.30<x<0$ or $1<x<1.30$.

25 $|x-2| \geqslant |2x+1|$

Since the expressions on both sides must be positive, we can square both sides and remove the absolute value signs.

$$(x-2)^2 \geqslant (2x+1)^2$$

$$x^2 - 4x + 4 \geqslant 4x^2 + 4x + 1$$

$$3x^2 + 8x - 3 \leqslant 0$$

$$x_{1,2} = \frac{-8 \pm \sqrt{8^2 - 4\cdot 3\cdot(-3)}}{2\cdot 3} = \frac{-8 \pm 10}{6}$$

$$x_1 = -3,\ x_2 = \frac{2}{6} = \frac{1}{3}$$

$$3(x+3)\left(x - \frac{1}{3}\right) \leqslant 0$$

$$(x+3)\left(x - \frac{1}{3}\right) \leqslant 0$$

We analyze the signs of both factors in a 'sign chart':

x		-3		$\frac{1}{3}$	
$x+3$	−	**0**	+		+
$x-\frac{1}{3}$	−		−	**0**	+
$(x+3)\left(x-\frac{1}{3}\right)$	+	**0**	−	**0**	+

The solution is: $-3 \leqslant x \leqslant \frac{1}{3}$.

26 For $f(x) = \frac{x+4}{x+1}$, $x \neq -1$ and $g(x) = \frac{x-2}{x-4}$, $x \neq 4$ we have:

$$f(x) \leqslant g(x)$$

$$\frac{x+4}{x+1} \leqslant \frac{x-2}{x-4}$$

$$\frac{x+4}{x+1} - \frac{x-2}{x-4} \leqslant 0$$

$$\frac{(x+4)(x-4)-(x-2)(x+1)}{(x+1)(x-4)} \leqslant 0$$

$$\frac{x^2 - 16 - x^2 + x + 2}{(x+1)(x-4)} \leqslant 0$$

$$\frac{x-14}{(x+1)(x-4)} \leqslant 0$$

The 'sign chart' for the inequality is:

x		-1		4		14	
$x+1$	−	**0**	+		+		+
$x-4$	−		−	**0**	+		+
$x-14$	−		−		−	**0**	+
$\frac{x-14}{(x+1)(x-4)}$	−	**X**	+	**X**	−	**0**	+

The solution is: $x < -1$ or $4 < x \leqslant 14$.

Solution Paper 1 type

27 $\left|\dfrac{x+9}{x-9}\right| \leqslant 2$

$$-2 \leqslant \frac{x+9}{x-9} \leqslant 2$$

We have two inequalities:

$$\frac{x+9}{x-9} \geqslant -2 \quad \text{and} \quad \frac{x+9}{x-9} \leqslant 2$$

$$\frac{x+9}{x-9} + 2 \geqslant 0 \qquad \frac{x+9}{x-9} - 2 \leqslant 0$$

$$\frac{3x-9}{x-9} \geqslant 0 \qquad \frac{-x+27}{x-9} \leqslant 0$$

$$\frac{x-3}{x-9} \geqslant 0 \qquad \frac{x-27}{x-9} \geqslant 0$$

We analyze the signs of the numerator and denominator for the first inequality in a 'sign chart':

x		3		9	
$x-3$	−	**0**	+		+
$x-9$	−		−	**0**	+
$\dfrac{x-3}{x-9}$	+	**0**	−	**X**	+

The first solution is: $x \leqslant 3$ or $x > 9$.

For the second inequality:

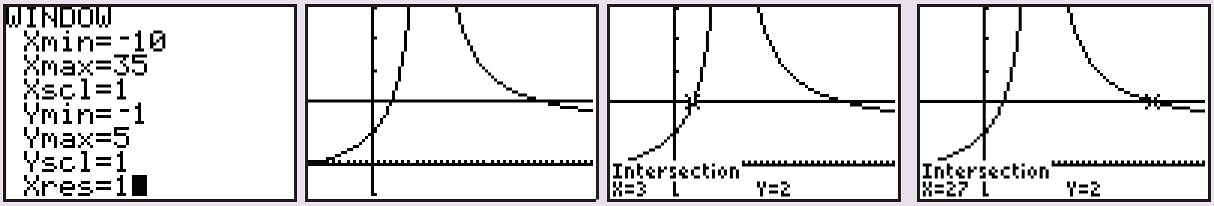
x		9		27	
$x-9$	−	**0**	+		+
$x-27$	−		−	**0**	+
$\dfrac{x-27}{x-9}$	+	**X**	−	**0**	+

The second solution is: $x < 9$ or $x \geqslant 27$.

The final solution is the intersection of the two solutions, i.e. $x \leqslant 3$ or $x \geqslant 27$.

Solution Paper 2 type

27



28 If $x_1 = 2 + i$ is the first root of the equation $x^3 - 6x^2 + 13x - 10 = 0$, then $x_2 = 2 - i$ is the second root, and the equation can be factorized as:

$$(x - (2 + i))(x - (2 - i))(x - a) = 0$$

$$((x - 2) - i)((x - 2) + i)(x - a) = 0$$

$$\left((x - 2)^2 - i^2\right)(x - a) = 0$$

$$\left(x^2 - 4x + 5\right)(x - a) = 0$$

$$x^3 - ax^2 - 4x^2 + 4ax + 5x - 5a = 0$$

$$x^3 + (-a - 4)x^2 + (4a + 5)x - 5a = 0$$

By comparing the coefficients with the coefficients of the original equation, we get:

$$-a - 4 = -6 \Rightarrow -a = -2 \Rightarrow a = 2$$

$$4a + 5 = 13 \Rightarrow 4a = 8 \Rightarrow a = 2$$

$$-5a = -10 \Rightarrow a = 2$$

We see that we get the solution $a = 2$ from all the equations, so the third root is $x_3 = 2$.

Solution Paper 1 type

29 The inequality $\frac{2x}{|x - 1|} < 1$ can be multiplied by the denominator since it is positive. Therefore: $2x < |x - 1|, x \neq 1$.

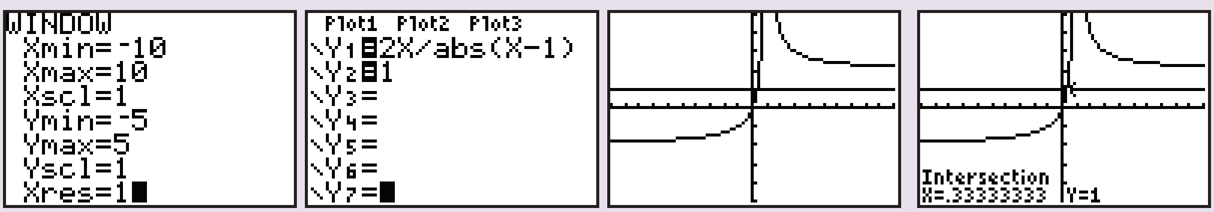
We have two possibilities:

If $x - 1 > 0$, i.e. $x > 1 \Rightarrow x - 1 > 2x \Rightarrow x < -1$, which is a contradiction.

If $x - 1 < 0$, i.e. $x < 1 \Rightarrow x - 1 > -2x \Rightarrow 3x < 1 \Rightarrow x < \frac{1}{3}$, which is a solution.

Solution Paper 2 type

29 For the inequality $\frac{2x}{|x - 1|} < 1$, we will observe where the graph of the function $f(x) = \frac{2x}{|x - 1|}$ is below the line $y = 1$.



The solution is: $x < \frac{1}{3}$.

Chapter 4

Practice questions

1 $a_1 = 4$

$a_4 = 19$

$\underline{a_n = 99}$

$$\left.\begin{aligned} 4 + 3d &= 19 \\ \underline{4 + (n-1)d} &\underline{= 99} \end{aligned}\right\} \Rightarrow d = 5 \text{ and } n = 20$$

2 $A = 3000, r = 0.06, n = 4, t = 6$

$$A = P\left(1+\frac{r}{n}\right)^{nt} \Rightarrow 3000 = P\left(1+\frac{0.06}{4}\right)^{4\cdot 6}$$

$$P = \frac{3000}{\left(1+\frac{0.06}{4}\right)^{4\cdot 6}} = 2098.63$$

You should invest €2098.63 now.

3 Nicks' studying hours form an arithmetic sequence with first term $a_1 = 12$ and common difference $d = 2$. Charlottes' studying hours form a geometric sequence with first term $b_1 = 12$ and common ratio $r = 1.1$.

a $a_5 = a_1 + 4d = 12 + 4 \cdot 2 = 20$

$b_5 = b_1 r^4 = 12 \cdot 1.1^4 \approx 17.57$

In week 5, Nick studied for 20 hours and Charlotte studied for 17.57 hours.

b $S_{arithmetic15} = \dfrac{15}{2}[2 \cdot 12 + (15-1)2] = 390$

$S_{geometric15} = 12\dfrac{1.1^{15}-1}{1.1-1} \approx 381.27$

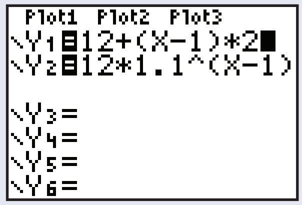
For the 15 weeks, Nick studied for a total of 390 hours and Charlotte studied for a total of 381.27 hours.

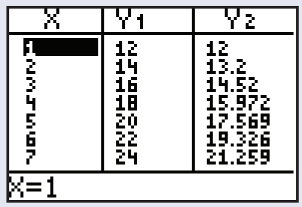
c $b_n > 40 \Rightarrow 12 \cdot 1.1^{n-1} > 40 \Rightarrow n > \dfrac{\log \frac{40}{12}}{\log 1.1} + 1 \Rightarrow$

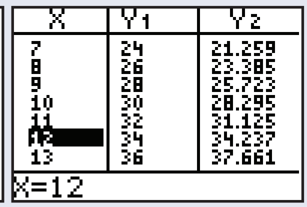
$n > 13.6$

Charlotte will exceed 40 hours of study per week in the 14th week.

d We need to determine n so that $b_n \geqslant a_n$. The easiest way of doing this is by entering n (as X), a_n (as Y_1), and b_n (as Y_2) into Table in a GDC.







We see that in week 12 Nick will study for 34 hours while Charlotte studies for 34.24 hours. So, Charlotte will catch up with Nick in week 12.

4 Plan A forms an arithmetic sequence with first term $a_1 = 1000$ and common difference $d = 80$. Plan B forms a geometric sequence with first term $b_1 = 1000$ and common ratio $r = 1.06$.

a $b_2 = 1000 \cdot 1.06 = 1060$ g

$b_3 = 1000 \cdot 1.06^2 = 1123.6$ g

b $a_{12} = a_1 + 11d = 1000 + 11 \cdot 80 = 1880$ g

$b_{12} = b_1 \cdot 1.06^{11} \approx 1898.3$ g

c **i** $S_{A12} = \dfrac{12}{2}(2a_1 + 11d) = 6(2 \cdot 1000 + 11 \cdot 80)$

$= 17\,280$ g

ii $S_{B12} = b_1 \dfrac{r^n - 1}{r - 1} = 1000\dfrac{1.06^{12} - 1}{1.06 - 1} \approx 16\,869.9$ g

5 **a** The initial amount forms a geometric sequence with $a_1 = 500$ and common ratio $r = 1.06$ (fixed rate 6% per annum).

After 10 years it will be worth:

$a_{11} = a_1 r^{10} = 500 \cdot 1.06^{10} = €895.42 = €895$ to the nearest euro.

b The future value is a partial sum of a geometric sequence:

$$FV = a_1\left(\frac{r^{11}-1}{r-1}-1\right) = 500\left(\frac{1.06^{11}-1}{1.06-1}-1\right)$$

$= €6985.82 = €6986$ to the nearest euro.

6 6, 9.5, 13, ... is an arithmetic sequence with $a_1 = 6$ and $d = 3.5$.

a $a_{40} = a_1 + 39d = 6 + 39\cdot 3.5 = 142.5$

b $S_{103} = \frac{103}{2}\left[2a_1 + (103-1)d\right]$

$= \frac{103}{2}(2\cdot 6 + 102\cdot 3.5)$

$= 19\,003.5$

7 For $a_n = \sqrt[3]{8 - a_{n-1}^3}$:

a $a_1 = 1, a_2 = \sqrt[3]{8-1^3} = \sqrt[3]{7}, a_3 = \sqrt[3]{8-(\sqrt[3]{7})^3} = 1$

b $a_1 = 2, a_2 = \sqrt[3]{8-2^3} = 0, a_3 = \sqrt[3]{8-0^3} = 2$

8 The training programme forms an arithmetic sequence with $a_1 = 2$ and $d = 0.5$.

a $a_n = 20 \Rightarrow 2 + (n-1)\cdot 0.5 = 20 \Rightarrow n = 37$

She first runs a distance of 20 km on the 37th day of her training.

b $S_{37} = \frac{37}{2}(2\cdot 2 + 36\cdot 0.5) = 407$

The total distance run during 37 days of training would be 407 km.

9 **a** $r = \frac{2400}{1600} = \frac{3600}{2400} = \frac{3}{2}$

b The number of new participants in 2012 is the 13th term in the sequence.

$a_{13} = a_1 r^{12} = 1600\cdot\left(\frac{3}{2}\right)^{12} = 207\ 594$

c $a_n > 50\,000 \Rightarrow 1600\left(\frac{3}{2}\right)^{n-1} > 50\,000 \Rightarrow$

$\left(\frac{3}{2}\right)^{n-1} > 31.25 \Rightarrow n > \frac{\log 31.25}{\log 1.5} + 1 \Rightarrow$

$n > 9.489$

The 10th term of the sequence will be greater than 50 000; therefore, the number of new participants will first exceed 50 000 in 2009.

d $S_{13} = a_1\frac{r^{13}-1}{r-1} = 1600\frac{1.5^{13}-1}{1.5-1} = 619\ 582$

e This trend in growth would not continue due to market saturation.

10 $a_1 = 25$

$a_4 = 13 \Rightarrow a_1 + 3d = 13 \Rightarrow 25 + 3d = 13 \Rightarrow d = -4$

$a_n = -11995 \Rightarrow a_1 + (n-1)d = -11995 \Rightarrow$

$25 + (n-1)(-4) = -11995 \Rightarrow n = 3006$

11 a $MN = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$

b $\text{Area}_{\square MNPQ} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$

c i $RS = \sqrt{\left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{8}+\frac{1}{8}} = \frac{1}{2}$

$\text{Area}_{\square RSTU} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

ii $1, \frac{1}{2}, \frac{1}{4}, \ldots \Rightarrow r = \frac{\frac{1}{2}}{1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

d i $\text{Area}_{10} = 1\left(\frac{1}{2}\right)^9 = \frac{1}{512}$

ii $S_\infty = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

12 Tim's swimming programme forms an arithmetic sequence with $a_1 = 200$ and $d = 20$.

a $a_{52} = a_1 + 51d = 200 + 51\cdot 20 = 1220$

Tim will swim 1220 metres in the final week.

b $S_{52} = \frac{52}{2}(2a_1 + 51d) = \frac{52}{2}(2\cdot 200 + 51\cdot 20) = 36\,920$

Altogether, Tim swims 36 920 metres.

13 a $\text{Area}_{\square A} = \left(\frac{3}{3}\right)^2 = 1$

$\text{Area}_{\square B} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

b $\text{Area}_{\square C} = \left(\frac{1}{9}\right)^2 = \frac{1}{81}$

c $\text{Shaded area}_2 = 1 + 8\cdot\frac{1}{9} = 1 + \frac{8}{9}$

$\text{Shaded area}_3 = \text{Shaded area}_2 + 8\cdot\frac{8}{81}$

$= 1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2$

d $\text{Shaded area}_\infty = 1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2 + \ldots = \frac{1}{1-\frac{8}{9}} = 9$

$\text{Unshaded area}_\infty = 9 - \text{Shaded area}_\infty = 0$

14 a i The series 2 + 22 + 222 + 2222 + ... is neither arithmetic nor geometric.

ii The series $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\ldots$ is geometric with $r=\frac{2}{3}<1$; thus converging.

iii The series $0.8+0.78+0.76+0.74+\ldots$ is arithmetic with $d=-0.02$.

iv The series $2+\frac{8}{3}+\frac{32}{9}+\frac{128}{27}+\ldots$ is geometric with $r=\frac{4}{3}>1$; thus diverging.

b For series **ii** we have: $S_\infty=\dfrac{2}{1-\frac{2}{3}}=6$.

15 The Kell scheme forms an arithmetic sequence with $a_1=18\,000$ and $d=400$. The IBO scheme forms a geometric sequence with $b_1=17\,000$ and $r=1.07$.

a All answers are in euros.

i Kell: $a_2=18\,000+400=18\,400$,

$a_3=a_2+400=18\,800$

IBO: $b_2=17\,000\cdot 1.07=18\,190$,

$b_3=b_2\cdot 1.07=19\,463.30$

ii Kell: $S_{10}=\frac{10}{2}(2\cdot 18\,000+9\cdot 400)$

$=198\,000$

IBO: $S_{10}=17\,000\,\dfrac{1.07^{10}-1}{1.07-1}$

$=234\,879.62$

iii Kell: $a_{10}=18\,000+9\cdot 400$

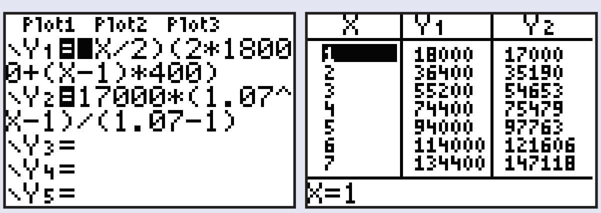
$=21600$

IBO: $b_{10}=17\,000\cdot 1.07^9$

$=31253.81$

b i From **a ii** we can see that $b_3>a_3$, so Merijayne will start earning more than Tim in the third year.

ii We can compare their total earnings with the help of a GDC. If X denotes the year, Y_1 represents total earnings for Tim and Y_2 for Merijayne, we have:



In the fourth year, Merijayne's total earnings will exceed those of Tim.

16 The number of seats in each row forms an arithmetic sequence with $a_1=16$ and $d=2$.

a $a_{24}=a_1+23d=16+23\cdot 2=62$

b $S_{24}=16+18+\ldots+62=\frac{24}{2}(16+62)=936$

17 The values of the investment after each year form a geometric sequence with $a_1=7000$, $r=1.0525$, and a_{n+1} represents the value of the investment after n years.

a Value of investment $=7000\cdot 1.0525^t$

b $7000\cdot 1.0525^t=10\,000 \Rightarrow 1.0525^t=\frac{10}{7}\Rightarrow$

$$t=\frac{\log\left(\frac{10}{7}\right)}{\log(1.0525)}=6.97$$

The minimum number of years is 7.

c If the rate of 5% is compounded quarterly, the value of the investment over 7 years would be:

$7000\cdot\left(1+\frac{5}{4\cdot 100}\right)^{7\cdot 4}=9911.95.$

For 5.25% compounded annually, the value of the investment would be $7000\cdot 1.0525^7=10\,015.04$.

Therefore, the investment at 5.25% annually is better.

18 a $S_1=9\Rightarrow a_1=9$

$S_2=20\Rightarrow a_1+a_2=20\Rightarrow 9+a_2=20\Rightarrow a_2=11$

b $d=a_2-a_1=11-9=2$

c $a_4=a_1+3d=9+3\cdot 2=15$

19 $\begin{cases}a_2=a+d=7\\ S_4=\frac{4}{2}[2a+3d]=12\end{cases}$

$\begin{cases}a+d=7\\ 2a+3d=6\end{cases}\Rightarrow a=15,\ d=-8$

20 $(1+x)^5(1+ax)^6=1+bx+10x^2+\quad+a^6x^{11}$

$(1+5x+10x^2+10x^3+5x^4+x^5)(1+6ax+15a^2x^2$
$+20a^3x^3+15a^4x^4+6a^5x^5+a^6x^6)$

$=1+bx+10x^2+\quad+a^6x^{11}$

$1+5x+6ax+10x^2+15a^2x^2+30ax^2+\quad+a^6x^{11}$

$=1+bx+10x^2+\quad+a^6x^{11}$

By comparing the coefficients of the same powers of x, we get:

$\begin{cases}5+6a=b\\ 10+15a^2+30a=10\end{cases}$

$a(a+2)=0\Rightarrow a=-2,\ b=-7$

21 $\begin{cases} a_5 : a_{12} = \dfrac{6}{13} \\ a_1 \cdot a_3 = 32 \end{cases}$

$\dfrac{a+4d}{a+11d} = \dfrac{6}{13} \Rightarrow a = 2d$

$a(a+2d) = 32 \Rightarrow 2d \cdot 4d = 32 \Rightarrow$

$d = 2$ (all terms positive), $a = 4$

$S_{100} = \dfrac{100}{2}(2 \cdot 4 + 99 \cdot 2) = 10\,300$

22 $22n - 3n - n = 18n = 9 \cdot 2n$ which is clearly divisible by 9 for $n = 1, 2, \ldots$.

23 $a_1 = 5$

$a_2 = a_1 + d = 13 \Rightarrow d = 8$

a $a_n = a_1 + (n-1)d = 5 + (n-1) \cdot 8 = 8n - 3$

b $a_n < 400 \Rightarrow 8n - 3 < 400 \Rightarrow n < 50.375$

There are 50 terms which are less then 400.

24 $(2+3x)^{10} = \sum_{i=0}^{10} \binom{10}{i} 2^{10-i}(3x)^i$

$i = 7 \Rightarrow \binom{10}{7} 2^{10-7}(3x)^7 = 120 \cdot 8 \cdot 2187x^7$

$= 2\,099\,520x^7$

The coefficient of x^7 is 2 099 520.

25 $S_n = 3n^2 - 2n \Rightarrow$

$S_1 = u_1 = 3 \cdot 1^2 - 2 \cdot 1 = 1$

$S_2 = u_1 + u_2 = u_1 + u_1 + d = 3 \cdot 2^2 - 2 \cdot 2 = 8 \Rightarrow$

$2 + d = 8 \Rightarrow d = 6$

$u_n = u_1 + (n-1)d = 1 + (n-1) \cdot 6 = 6n - 5$

26 Six people can be ordered in 6! ways, but, as they are seated around a circular table, all circular permutations that come in groups of six (ABCDEF, BCDEFA, CDEFAB,…) are equivalent, so there are actually $\dfrac{6!}{6}$ ways. As Mr Black and Mrs White should not sit together, we must subtract all circular permutations in which this pair is regarded as one person, but multiplied by 2, because a male can be on the left or right of the female. Altogether, there are $\dfrac{6!}{6} - \dfrac{5!}{5} \cdot 2 = 120 - 24 \cdot 2 = 72$ ways.

27 Firstly, we must determine which is the last positive term in the sequence.

$a_1 = 85, d = -7$

$a_n = 85 + (n-1)(-7) > 0 \Rightarrow 92 - 7n > 0 \Rightarrow n < 13.14$

For $n = 13 \Rightarrow S_{13} = \dfrac{13}{2}[2 \cdot 85 + 12 \cdot (-7)] = 559$.

28 $\left(x + \dfrac{1}{ax^2}\right)^7 = \sum_{i=0}^{7} \binom{7}{i} x^{7-i} \left(\dfrac{1}{ax^2}\right)^i$

$i = 2 \Rightarrow \binom{7}{2} x^{7-2} \left(\dfrac{1}{ax^2}\right)^2 = 21x^5 \cdot \dfrac{1}{a^2x^4} = \dfrac{21x}{a^2}$

$\dfrac{21}{a^2} = \dfrac{7}{3} \Rightarrow a^2 = 9 \Rightarrow a = 3,\ \cancel{a = -3}$

29 $S_\infty = \dfrac{a}{1-r} = \dfrac{27}{2} \Rightarrow a = \dfrac{27(1-r)}{2}$

$S_3 = a\dfrac{1-r^3}{1-r} = 13$

$\dfrac{27\cancel{(1-r)}}{2} \cdot \dfrac{1-r^3}{\cancel{1-r}} = 13 \Rightarrow r^3 = \dfrac{1}{27} \Rightarrow r = \dfrac{1}{3}, a = 9$

30 Student A can get 1 coin in $\binom{6}{1}$ ways.

Student A can get 2 coins in $\binom{6}{2}$ ways.

Student A can get 3 coins in $\binom{6}{3}$ ways.

Student A can get 4 coins in $\binom{6}{4}$ ways.

Student A can get 5 coins in $\binom{6}{5}$ ways.

Altogether, there are

$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} = 6 + 15 + 20 + 15 + 6 = 62$

ways.

31 This is an infinite geometric series with:

$a_1 = -12, r = -\dfrac{2}{3}$. So:

$S_\infty = \dfrac{a_1}{1-r} = \dfrac{-12}{1+\dfrac{2}{3}} = \dfrac{-36}{5}$.

32 For $u_n = 3(4)^{n+1}, n \in \mathbb{Z}^+$:

a $u_1 = 48, r = 4$

b $S_n = 48\dfrac{4^n - 1}{4 - 1} = 16(4^n - 1)$

33 For the infinite geometric series

$1 + \left(\dfrac{2x}{3}\right) + \left(\dfrac{2x}{3}\right)^2 + \left(\dfrac{2x}{3}\right)^3 + \ldots$:

a $a_1 = 1, r = \dfrac{2x}{3}$

The series converges for

$|r| < 1 \Rightarrow \left|\dfrac{2x}{3}\right| < 1 \Rightarrow -\dfrac{3}{2} < x < \dfrac{3}{2}$.

b $x = 1.2 \Rightarrow r = \dfrac{2 \cdot 1.2}{3} = \dfrac{4}{5}$

$$S_\infty = \frac{1}{1-\frac{4}{5}} = 5$$

34 There are $9999 - 999 = 9000$ four-digit numbers.

Without digit 3 there are $8 \cdot 9 \cdot 9 \cdot 9 = 5832$ numbers.

So, with at least one digit 3, there are $9000 - 5832 = 3168$ numbers.

35 For the arithmetic series we have: $a_1 = 2, d = 3$.

a $S_n = \dfrac{n}{2}[2 \cdot 2 + (n-1) \cdot 3] = \dfrac{n(3n+1)}{2}$

b $S_n = 1365 \Rightarrow \dfrac{n(3n+1)}{2} = 1365 \Rightarrow$

$3n^2 + n - 2730 = 0 \Rightarrow$

$$n = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 3 \cdot (-2730)}}{2 \cdot 3} = 30, \cancel{-\frac{91}{3}}$$

36 $$(1 - \tfrac{1}{2}x)^8 = \sum_{i=0}^{8} \binom{8}{i} \left(-\frac{1}{2}x\right)^i$$

$$i = 3 \Rightarrow \binom{8}{3}\left(-\frac{1}{2}x\right)^3 = 56\left(-\frac{1}{8}x^3\right) = -7x^3$$

The coefficient of x^3 is -7.

37 $$\sum_{r=1}^{50} \ln(2^r) = \sum_{r=1}^{50} r\ln(2) = \ln 2 \sum_{r=1}^{50} r$$

$$= (\ln 2)\left[\frac{50}{2}(2 + 49)\right]$$

$$= 1275 \ln 2$$

38 For $u_0 = 1, u_1 = 2, u_{n+1} = 3u_n - 2u_{n-1}, n \in \mathbb{Z}^+$:

a $u_2 = 3u_1 - 2u_0 = 4$

$u_3 = 3u_2 - 2u_1 = 8$

$u_4 = 3u_3 - 2u_2 = 16$

b **i** $u_n = 2^n$

ii $$\begin{aligned} 3u_n - 2u_{n-1} &= 3 \cdot 2^n - 2 \cdot 2^{n-1} \\ &= 3 \cdot 2^n - 2^n \\ &= 2 \cdot 2^n \\ &= 2^{n+1} = u_{n+1} \end{aligned}$$

39 **a** $\begin{cases} S_2 = a + ar = 15 \\ S_\infty = \dfrac{a}{1-r} = 27 \end{cases}$

$\begin{cases} a(1+r) = 15 \\ a = 27(1-r) \end{cases} \Rightarrow$

$27(1-r)(1+r) = 15 \Rightarrow 27(1-r^2) = 15 \Rightarrow$

$r^2 = \dfrac{4}{9} \Rightarrow r = \dfrac{2}{3}$ (all terms positive)

b $a = 27\left(1 - \dfrac{2}{3}\right) = 9$

40 For the arithmetic sequence $2, a-b, 2a+b+7, a-3b$, the difference must be constant.

So, we have the following system of equations:

$$\begin{cases} (a-b) - 2 = (2a+b+7) - (a-b) \\ (a-3b) - (2a+b+7) = (2a+b+7) - (a-b) \end{cases}$$

$$\begin{cases} a - b - 2 = a + 2b + 7 \\ -a - 4b - 7 = a + 2b + 7 \end{cases}$$

$$\begin{cases} 3b = -9 \\ a + 3b = -7 \end{cases} \Rightarrow a = 2, b = -3$$

41 Let A and B denote the two oldest children that cannot both be chosen. Four children can be then be chosen as 4 out of 6 (without A and B), or as 3 out of 6 and 1 out of 2 (A or B). Altogether, there are

$\dbinom{6}{4} + \dbinom{6}{3}\dbinom{2}{1} = 15 + 20 \cdot 2 = 55$ ways.

42 As $a, 1, b$ form an arithmetic progression, the difference is the same: $1 - a = b - 1$.

As $1, a, b$ form a geometric progression, the ratio is the same: $\dfrac{a}{1} = \dfrac{b}{a}$.

We solve the following system of equations:

$$\begin{cases} 1 - a = b - 1 \\ \dfrac{a}{1} = \dfrac{b}{a} \end{cases}$$

$$\begin{cases} a + b = 2 \\ a^2 = b \end{cases} \Rightarrow a^2 + a - 2 = 0 \Rightarrow (a-1)(a+2) = 0$$

$\Rightarrow a = 1$ or $a = -2$

As $a = 1$ gives $b = 1$, the solution is $a = -2, b = 4$.

43 We can see that:

$OB = OA = 1$

$\cos\theta = \dfrac{OB_1}{OA} \Rightarrow OB_1 = OA_1 = OA\cos\theta = \cos\theta$

$\cos\theta = \dfrac{OB_2}{OA_1} \Rightarrow OB_2 = OA_2 = OA_1\cos\theta = \cos^2\theta$

$\cos\theta = \dfrac{OB_3}{OA_2} \Rightarrow OB_3 = OA_3 = OA_2\cos\theta = \cos^3\theta$

The radii form the geometric sequence
$1, \cos\theta, \cos^2\theta, \cos^3\theta, \ldots$

As the length of the arc is equal to $\theta \cdot$ radius, the sum of the arc lengths is:

$AB + A_1B_1 + A_2B_2 + A_3B_3 + \ldots =$

$$\theta + \theta\cos\theta + \theta\cos^2\theta + \theta\cos^3\theta + \ldots = \frac{\theta}{1-\cos\theta}$$

44 For $S_n = 2n^2 - n, n \in \mathbb{Z}^+$, we have:

a $S_1 = u_1 = 2\cdot 1^2 - 1 = 1$

$S_2 = u_1 + u_2 = 2\cdot 2^2 - 2 = 6 \Rightarrow$

$u_1 + u_2 = 6 \Rightarrow 1 + u_2 = 6 \Rightarrow u_2 = 5$

$S_3 = u_1 + u_2 + u_3 = 2\cdot 3^2 - 3 = 15 \Rightarrow$

$u_1 + u_2 + u_3 = 15 \Rightarrow 1 + 5 + u_3 = 15 \Rightarrow u_3 = 9$

b $u_n = S_n - S_{n-1} = 2n^2 - n - \left[2(n-1)^2 - (n-1)\right]$

$= 2n^2 - n - \left[2n^2 - 4n + 2 - n + 1\right]$

$= 4n - 3$

45 a $(2+x)^5 = \sum_{i=0}^{5}\binom{5}{i}2^{5-i}x^i =$

$\binom{5}{0}2^{5-0}x^0 + \binom{5}{1}2^{5-1}x^1 + \binom{5}{2}x^{5-2}x^2 + \binom{5}{3}2^{5-3}x^3$

$+\binom{5}{4}2^{5-4}x^4 + \binom{5}{5}2^{5-5}x^5$

$= 32 + 5\cdot 16x + 10\cdot 8x^2 + 10\cdot 4x^3 + 5\cdot 2x^4 + x^5$

$= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$

b $2.01^5 = (2+0.01)^5 = 32 + 80\cdot 0.01 + 80\cdot 0.01^2$

$+ 40\cdot 0.01^3 + 10\cdot 0.01^4 + 0.01^5 = 32 + 0.8 + 0.008$

$+ 0.000\,04 + 0.000\,0001 + 0.000\,000\,0001$

$= 32.808\,040\,1001$

46 The interest is paid yearly. For n years, rate $r = 0.063$, and principal $P = 5000$ after n full years:

a $A_n = P(1+r)^n = 5000\,(1.063)^n$

b $A_5 = 5000\,(1.063)^5 = 6786.35$

c **i** $5000\,(1.063)^n > 10\,000$

ii $(1.063)^n > 2 \Rightarrow n > \dfrac{\log 2}{\log 1.063} = 11.35$

The value will double after 12 full years.

47 Let $S(n)$ be the statement: $5^n + 9^n + 2$ is divisible by 4, for $n \in \mathbb{Z}^+$.

Basis step:

$S(1)$:

$S_n = 4n^2 - 2n$

$5^1 + 9^1 + 2 = 16$ which is divisible by 4.

Inductive step:

Assume $S(k)$ is true, i.e. assume that $5^k + 9^k + 2$ is divisible by 4. So, we assume that

$5^k + 9^k + 2 = 4A, A \in \mathbb{Z}$ (*) $\Rightarrow 9^k = 4A - 5^k - 2$.

Then $S(k+1)$:

$5^{k+1} + 9^{k+1} + 2 = 5^{k+1} + 9\cdot(4A - 5^k - 2) + 2$

$= 5\cdot 5^k + 36A - 9\cdot 5^k - 18 + 2$

$= 36A - 4\cdot 5^k - 16$

$= 4(9A - 5^k - 4)$,

i.e. $5^k + 9^k + 2$ is divisible by 4.

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $5^n + 9^n + 2$ is divisible by 4, for $n \in \mathbb{Z}^+$.

48 For an arithmetic sequence with $S_n = 4n^2 - 2n$ we have:

$u_2 = S_2 - S_1 = (4\cdot 2^2 - 2\cdot 2) - (4\cdot 1^2 - 2\cdot 1)$

$= 12 - 2 = 10$

$u_m = S_m - S_{m-1} = \left(4m^2 - 2m\right) - \left[4(m-1)^2 - 2(m-1)\right]$

$= 4m^2 - 2m - \left[4m^2 - 8m + 4 - 2m + 2\right]$

$= 8m - 6$

$u_{32} = S_{32} - S_{31} = (4\cdot 3 = 2^2 - 2\cdot 32) - (4\cdot 31^2 - 2\cdot 31)$

$= 250$

As they are consecutive terms in a geometric sequence, the ratio is the same.

$\dfrac{8m-6}{10} = \dfrac{250}{8m-6} \Rightarrow$

$(8m-6)^2 = 2500 \Rightarrow$

$8m - 6 = 50 \Rightarrow m = 7$

Chapter 5

Practice questions

1 **a** To find the x-coordinate of P we have to solve the equation:

$2-\log_3(x+1)=0 \Rightarrow \log_3(x+1)=2 \Rightarrow x+1=3^2 \Rightarrow x=8$. So, the point is $P(8, 0)$ and it's x-coordinate is 8.

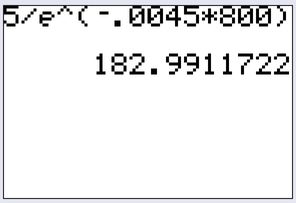
b We will find the y-coordinate of Q by finding the value of the function for $x=0$:

$2-\log_3(0+1)=0 \Rightarrow 2-\log_3 1=2$; hence, the y-coordinate of Q is 2.

c The y-coordinate of R is 3. To find the x-coordinate of Q we have to solve the equation:

$2-\log_3(x+1)=3 \Rightarrow \log_3(x+1)=-1 \Rightarrow x+1=3^{-1} \Rightarrow x=-\frac{2}{3}$. Hence, the point is $R\left(-\frac{2}{3}, 3\right)$.

2 **a** We have to solve the equation $A(800)=5$. Hence, $5=A_0e^{-0.0045\cdot 800} \Rightarrow A_0=\frac{5}{e^{-3.6}}\approx 183$.



So, approximately 183 grams were present initially.

b We have to solve the equation $A(t)=\frac{A_0}{2}$. Hence,

$$\frac{A_0}{2}=A_0e^{-0.0045\cdot t} \Rightarrow \frac{1}{2}=e^{-0.0045\cdot t} \Rightarrow \ln\left(\frac{1}{2}\right)=-0.0045t \Rightarrow t=\frac{\ln\left(\frac{1}{2}\right)}{-0.0045}\approx 154.$$

So, the half-life of the substance is approximately 154 years.

Note: We could have started by using the result from **a** such that we would have to solve: $\frac{183}{2}=183e^{-0.0045\cdot t}$. However, it is better, if possible, to use only the data given in the question text. In this way, possible mistakes made in previous calculations are not carried through a problem.

3 **a** Using the properties of logarithms, we can write the terms of the sequence differently:

$\ln y+\ln y^2+\ln y^3+\ldots=\ln y+2\ln y+3\ln y+\ldots$. Now, we can see that the sequence is arithmetic with first term $u_1=\ln y$, and common difference $d=2\ln y-\ln y=\ln y$. Hence, $u_n=\ln y+(n-1)\ln y$. Using the property of logarithms, we can simplify this expression: $u_n=\ln y+(n-1)\ln y=1\cdot\ln y+(n-1)\ln y=(1+n-1)\ln y=n\ln y$.

Sum of $S_n=\frac{n}{2}\left(\ln y+\ln y^n\right)=\frac{n}{2}\ln\left(yy^n\right)=\frac{n}{2}\ln y^{n+1}=\frac{n}{2}(n+1)\ln y$. So, $S_n=\frac{n(n+1)}{2}\ln y$.

b Using the properties of logarithms, we can write the terms of the sequence differently:

$\ln(xy)+\ln\left(xy^2\right)+\ln\left(xy^3\right)+\ldots=(\ln x+\ln y)+(\ln x+2\ln y)+(\ln x+3\ln y)+\ldots$

We can see that each term of this sequence matches the corresponding term of the sequence in part **a** plus

$\ln x$. Hence, $u_n=\ln x+\ln y^n$, and the sum: $S_n=\sum_{i=1}^{n}\left(\ln x+\ln y^i\right)=\sum_{i=1}^{n}\ln x+\sum_{i=1}^{n}\ln y^i=n\ln x+\frac{n(n+1)}{2}\ln y$.

4 Using the properties of logarithms, we can transform the equation:

$\log_2\left(5x^2-x-2\right)=2+2\log_2 x \Rightarrow \log_2\left(5x^2-x-2\right)=\log_2 4+\log_2 x^2 \Rightarrow \log_2\left(5x^2-x-2\right)=\log_2 4x^2$

$5x^2-x-2=4x^2 \Rightarrow x^2-x-2=0 \Rightarrow x_1=-1, x_2=2$

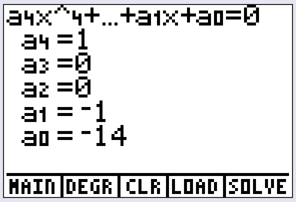
Since, $\log_2 x$ is not defined for -1, the only solution is $x=2$.

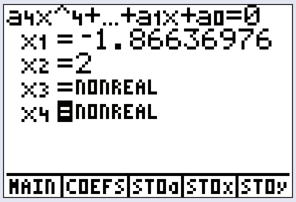
5 We can determine the value of x: $x = \log_2 4\sqrt{2} = \log_2 2^{2+\frac{1}{2}} = \log_2 2^{\frac{5}{2}} = \frac{5}{2}$.

From $\log_z y = 4$ we can conclude that: $y = z^4$.

From the third equation, we can conclude: $z^4 = 4\cdot\left(\frac{5}{2}\right)^2 - 2\frac{5}{2} - 6 + z = 25 - 5 - 6 + z = 14 + z \Rightarrow z^4 - z - 14 = 0$

We will solve this equation using the PolySmlt/PolyRootFinder application:





From $\log_z y = 4$, we can conclude that the base of the logarithm should be positive; hence, $z = 2$, and thus $y = z^4 = 16$.

6 Transforming the equation: $2e^{3t} - 7e^{2t} + 7e^t = 2 \Rightarrow 2e^{3t} - 7e^{2t} + 7e^t - 2 = 0$

Grouping the terms with the same coefficients: $2\left(e^{3t} - 1\right) - 7\left(e^{2t} - e^t\right) = 0 \Rightarrow 2\left(e^{3t} - 1\right) - 7e^t\left(e^t - 1\right) = 0$

Using the formula for the difference of cubes:

$2\left(e^t - 1\right)\left(e^{2t} + e^t + 1\right) - 7e^t\left(e^t - 1\right) = 0 \Rightarrow \left(e^t - 1\right)\left(2e^{2t} + 2e^t + 2 - 7e^t\right) = 0 \Rightarrow \left(e^t - 1\right)\left(2e^{2t} - 5e^t + 2\right) = 0$

Hence, either $e^t - 1 = 0 \Rightarrow e^t = 1 \Rightarrow t = 0$, or $2e^{2t} - 5e^t + 2 = 0 \Rightarrow 2\left(e^t\right)^2 - 5e^t + 2 = 0$.

We can solve the second equation by using a substitution: $e^t = s \Rightarrow 2s^2 - 5s + 2 = 0 \Rightarrow s_1 = \frac{1}{2}, s_2 = 2$; therefore: $e^t = \frac{1}{2} \Rightarrow t = \ln\frac{1}{2}$, or $e^t = 2 \Rightarrow t = \ln 2$.

Solutions are: $t = 0, \ln\frac{1}{2}, \ln 2$.

7 Using the substitution $\ln x = t$:

$$8e^2 - 2et = t^2 \Rightarrow t^2 + 2et - 8e^2 \Rightarrow t_{1,2} = \frac{-2e \pm \sqrt{4e^2 + 32e^2}}{2} = \frac{-2e \pm 6e}{2} \Rightarrow t_1 = -4e, t_2 = 2e$$

Hence:

$\ln x = -4e \Rightarrow x = e^{-4e}$

$\ln x = 2e \Rightarrow x = e^{2e}$

8 **a** Using the change of base formula: $\log_3 x - 4\frac{\log_3 3}{\log_3 x} + 3 = 0 \Rightarrow \log_3 x - 4\frac{1}{\log_3 x} + 3 = 0$

Multiplying by $\log_3 x$: $(\log_3 x)^2 - 4 + 3\log_3 x = 0$

Using the substitution $\log_3 x = t$: $t^2 + 3t - 4 = 0 \Rightarrow t_1 = -4, t_2 = 1$

Therefore, the solutions are:

$\log_3 x = -4 \Rightarrow x = 3^{-4} = \frac{1}{81}$

$\log_3 x = 1 \Rightarrow x = 3$

We can see from the original equation that x has to be positive and $x \neq 1$, so $x = \frac{1}{81}$ and $x = 3$ are both solutions.

b Using the properties of logarithms:

$\log_2(x-5) + \log_2(x+2) = 3 \Rightarrow \log_2(x-5)(x+2) = 3 \Rightarrow (x-5)(x+2) = 2^3 \Rightarrow x^2 - 3x - 10 = 8$

The solutions of the quadratic equation are 6 and -3. Since x has to be greater than 5 (for $\log_2(x-5)$ and $\log_2(x+2)$ to be defined), the only solution is $x = 6$.

9 a First using the property $k \log_b M = \log_b (M^k)$ and then the properties of the sum and difference of logarithms, we have:

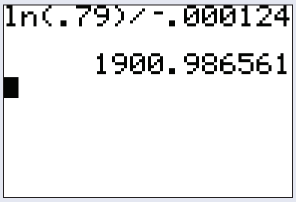
$$2\log a + 3\log b - \log c = \log a^2 + \log b^3 - \log c = \log(a^2b^3) - \log c = \log\frac{a^2b^3}{c}.$$

b First using the property $k \log_b M = \log_b (M^k)$ and then $\ln e = 1$, and, finally, the properties of the sum and difference of logarithms, we have:

$$3\ln x - \frac{1}{2}\ln y + 1 = \ln x^3 - \ln y^{\frac{1}{2}} + \ln e = (\ln x^3 + \ln e) - \ln\sqrt{y} = \ln(ex^3) - \ln\sqrt{y} = \ln\frac{ex^3}{\sqrt{y}}.$$

10 Given that $A(t) = 0.79A_0$, we have to solve the equation: $0.79A_0 = A_0e^{-0.000124t}$. So, we have:

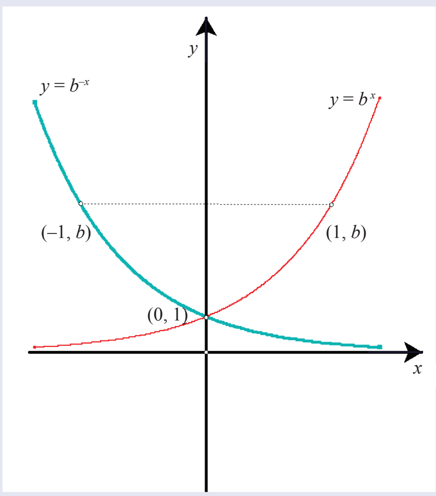
$$0.79 = e^{-0.000124t} \Rightarrow \ln 0.79 = -0.000\,124t \Rightarrow t = \frac{\ln 0.79}{-0.000\,124} \approx 1900 \text{ years.}$$



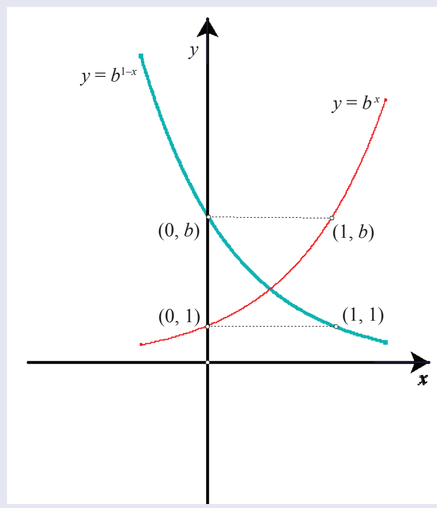
11 c is solution of the equation: $0 = \log_3(2x-3) - 4 \Rightarrow \log_3(2x-3) = 4 \Rightarrow 2x - 3 = 3^4 \Rightarrow 2x = 81 + 3 \Rightarrow x = 42$

So, $c = 42$.

12 a



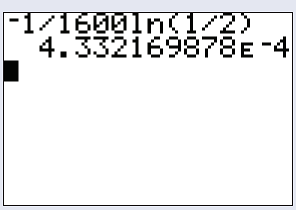
b



13 a Since its half-life is 1600 years, the exponential decay model is: $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{1600}}$. So, we have to solve:

$A_0\left(\frac{1}{2}\right)^{\frac{t}{1600}} = A_0e^{-kt}$. Since the equation holds for all t, then:

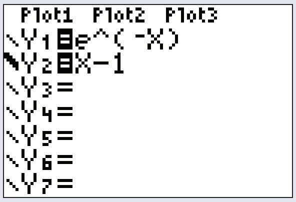
$$\left(\frac{1}{2}\right)^{\frac{1}{1600}} = e^{-k} \Rightarrow \ln\left(\frac{1}{2}\right)^{\frac{1}{1600}} = -k \Rightarrow k = -\frac{1}{1600}\ln\left(\frac{1}{2}\right) \approx 0.000\,4332 \text{ (4 s.f.)}$$

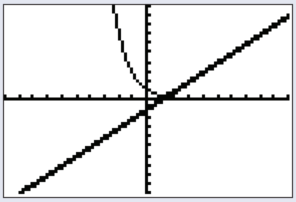


b We have to find: $\dfrac{A(4000)}{A_0} = \dfrac{A_0\left(\frac{1}{2}\right)^{\frac{4000}{1600}}}{A_0} = \left(\dfrac{1}{2}\right)^{2.50} = 0.176\,7766... \approx 17.7\%$

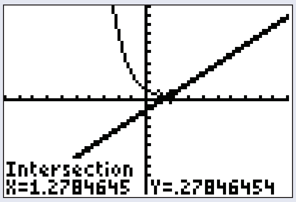
Note: We would get the same result if we used $A(t) = A_0 e^{-0.0004332t}$. Using the result from **a**, we have: $e^{-0.0004332 \cdot 4000} \approx e^{-1.7328} \approx 17.7\%$.

14 This equation cannot be solved exactly, so we will use a GDC. We transform the equation: $e^{-x} = x - 1$. We know the behaviour of both functions, so we can determine the number of solutions on the graph, and hence find all the solutions.



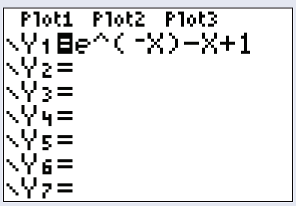


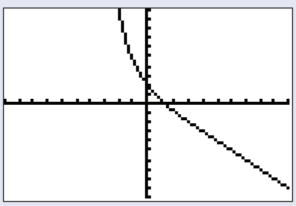
So, there is only one solution:



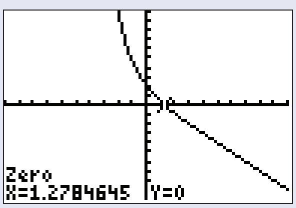
Solution is: $x \approx 1.28$.

Note: If we graph the original equation to find the zeros, we can't be sure that our window is good enough to find all the values.

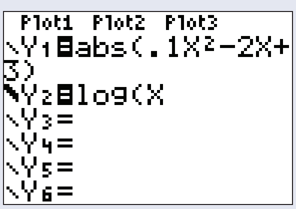


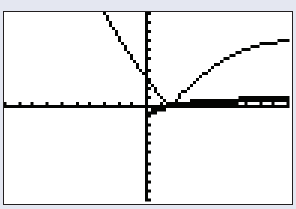


In this case, we will find the solution:

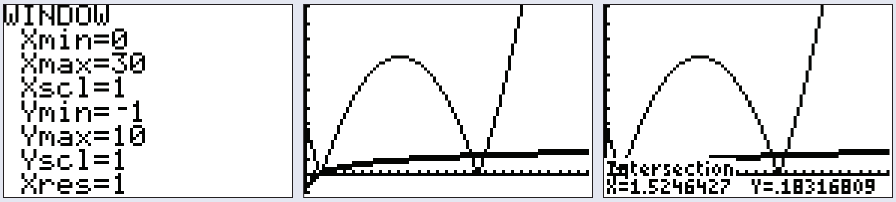


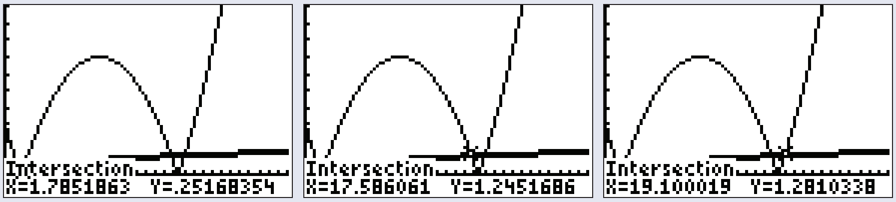
15 This inequality cannot be solved exactly, so we will use a GDC.





We have to change the window to see all the regions where the logarithmic function is above the graph of the absolute value. We will take into account that the logarithm is defined for positive values, and that the equation $0.1x^2 - 2x + 3 = 0$ has two positive solutions ($x_1 \approx 1.64$, $x_2 \approx 18.36$), so the absolute value function will decrease to zero once again.

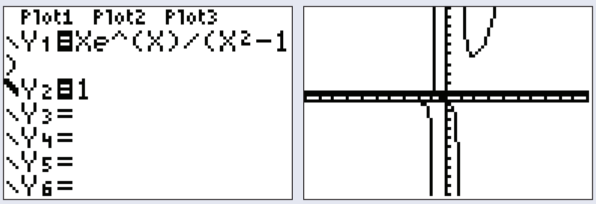




So, the first interval is $1.52 < x < 1.79$ and the second is $17.6 < x < 19.1$.

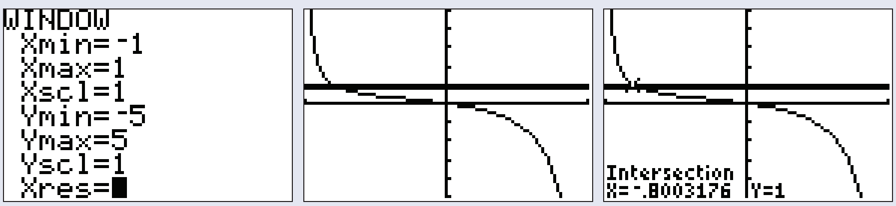
Hence, the solution set is: $]1.52, 1.79[\ \cup\ ]17.6, 19.1[$

16 This inequality cannot be solved exactly, so we will use a GDC.



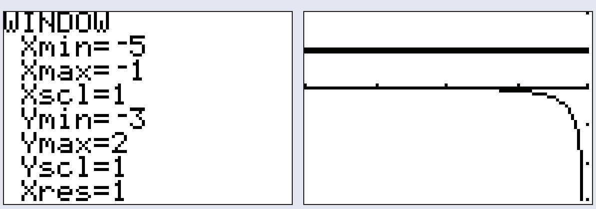
The function $f(x) = \dfrac{xe^x}{x^2 - 1}$ has vertical asymptotes $x = -1$, $x = 1$; so, we have to explore three regions. For $x > 1$, the graph of $y = f(x)$ is above the line $y = 1$; hence, $x > 1$ is in the solution set.

For $-1 < x < 1$, we have:



So, for $-1 < x < -0.800$, the graph of $y = f(x)$ is above the line $y = 1$; hence, $-1 < x < -0.800$ is in the solution set.

For $x < -1$, the graph of $y = f(x)$ is below the line $y = 1$; hence, there are no solutions for $x < -1$.



So, the solution set is: $]-1, -0.800[\ \cup\ ]1, +\infty[$

17 a Transforming the equation: $2\left(4^x\right) + 4^{-x} = 3 \Rightarrow 2\left(4^x\right) + \dfrac{1}{4^x} = 3 \Rightarrow 2\left(4^x\right)^2 - 3\left(4^x\right) + 1 = 0$

Using the substitution $4^x = t$: $2t^2 - 3t + 1 = 0 \Rightarrow t_1 = \dfrac{1}{2}, t_2 = 1$

Hence,

$4^x = \frac{1}{2} \Rightarrow 4^x = 4^{-\frac{1}{2}} \Rightarrow x = -\frac{1}{2}$

$4^x = 1 \Rightarrow x = 0$

b **i** $a^x = e^{2x+1} \Rightarrow \ln a^x = 2x+1 \Rightarrow x\ln a - 2x = 1 \Rightarrow x(\ln a - 2) = 1 \Rightarrow x = \frac{1}{\ln a - 2}$

Note: Instead of taking ln from both sides of the equation, we can take $\log_a$. Then we will have:

$a^x = e^{2x+1} \Rightarrow x = \log_a e^{(2x+1)} \Rightarrow x = (2x+1)\log_a e \Rightarrow x - 2x\log_a e = \log_a e$

$\Rightarrow x(1 - 2\log_a e) = \log_a e \Rightarrow x = \frac{\log_a e}{1 - 2\log_a e}$

ii The equation will not have a solution when $\frac{\log_a e}{1 - 2\log_a e}$ is not defined. Hence,

$1 - 2\log_a e = 0 \Rightarrow \log_a e = \frac{1}{2} \Rightarrow e = a^{\frac{1}{2}} \Rightarrow e^2 = a$

18 Transforming the equation:

$2^{2x+3} = 2^{x+1} + 3 \Rightarrow 2^3 \cdot 2^{2x} - 2 \cdot 2^x - 3 = 0 \Rightarrow 8 \cdot (2^x)^2 - 2 \cdot 2^x - 3 = 0$

Using the substitution $2^x = t$: $8 \cdot t^2 - 2 \cdot t - 3 = 0 \Rightarrow t_1 = -\frac{1}{2}, t_3 = \frac{3}{4}$

Hence,

$2^x = -\frac{1}{2}$ has no solution

$2^x = \frac{3}{4} \Rightarrow x = \log_2 \frac{3}{4} = \log_2 3 - \log_2 4 = \log_2 3 - 2$; so, $a = -2$ and $b = 3$.

19 Using the substitution $\ln x = t$: $2t^2 = 3t - 1 \Rightarrow 2t^2 - 3t + 1 = 0 \Rightarrow t_1 = \frac{1}{2}, t_2 = 1$. Hence, the solutions are:

$\ln x = \frac{1}{2} \Rightarrow x = e^{\frac{1}{2}} = \sqrt{e}$

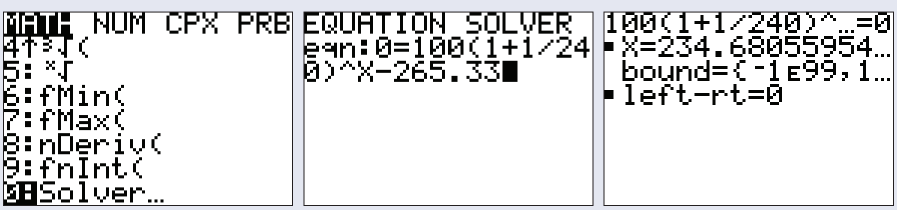
$\ln x = 1 \Rightarrow x = e$

20 **a** We use the exponential function for compound interest: $V(t) = 100(1 + 0.05)^t$. The value after 20 years is: $V(20) = 100(1.05)^{20} = 265.33$ (dollars).

b We use the exponential function for compound interest: $A(t) = 100\left(1 + \frac{\frac{5}{12}}{100}\right)^t$, where time is given in months. So, we have to solve $A(t) > 265.33$.

$100\left(1 + \frac{1}{240}\right)^t > 265.33 \Rightarrow \left(\frac{241}{240}\right)^t = 2.6533 \Rightarrow t\log\frac{241}{240} > \log 2.6533$. Dividing the inequality by $\log\frac{241}{240}$ (which is positive), we have: $t > \frac{\log 2.6533}{\log\frac{241}{240}} \approx 234.68$. So, the value of the investment will exceed V after 235 months.

Note: Since we are not looking for the exact solution, we can solve the inequality using a GDC. We are looking for the smallest value, so we will use Math/Solver.



This process is much quicker than solving algebraically.

21 We will use the change of base formula:

$$9\log_5 x = 25\frac{\log_5 5}{\log_5 x} \Rightarrow 9\log_5 x = 25\frac{1}{\log_5 x} \Rightarrow 9\left(\log_5 x\right)^2 = 25 \Rightarrow \left(\log_5 x\right)^2 = \frac{25}{9}$$

Hence, the solutions are:

$$\log_5 x = -\frac{5}{3} \Rightarrow x = 5^{-\frac{5}{3}}$$

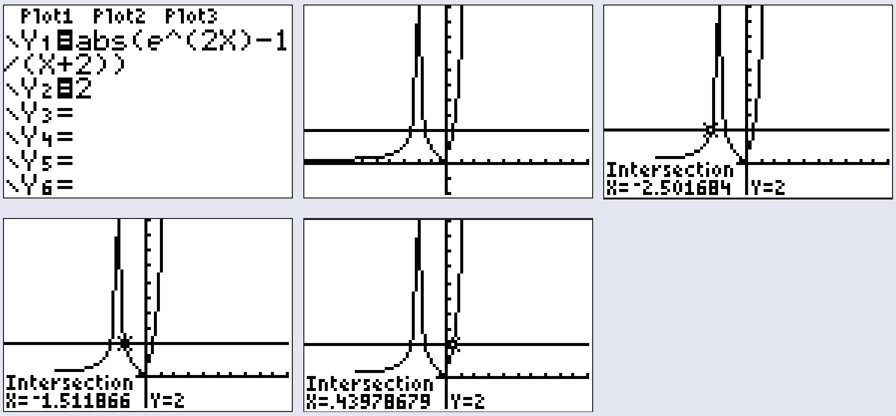
$$\log_5 x = \frac{5}{3} \Rightarrow x = 5^{\frac{5}{3}}$$

22 From $\left|\ln(x+3)\right| = 1$ we can conclude that either $\ln(x+3) = -1$, or $\ln(x+3) = 1$. Hence,

$$\ln(x+3) = -1 \Rightarrow x+3 = e^{-1} \Rightarrow x = e^{-1} - 3 = \frac{1}{e} - 3$$

$$\ln(x+3) = 1 \Rightarrow x+3 = e^{1} \Rightarrow x = e - 3$$

23 Since the equation cannot be solved algebraically, we will use a GDC.



So, the solutions are: $-2.50, -1.51, 0.440$.

Note: We can transform the equation and reduce the task by solving two equations graphically: $e^{2x} - \dfrac{1}{x+2} = -2$ and $e^{2x} - \dfrac{1}{x+2} = 2$.

24 Since the number of bacteria double every 20 minutes, the exponential growth model is: $n = 650(2)^{\frac{t}{20}}$. So, we have to solve: $650(2)^{\frac{t}{20}} = 650e^{kt}$. Since the equation holds for all t, then $(2)^{\frac{1}{20}} = e^{k} \Rightarrow \ln(2)^{\frac{1}{20}} = k \Rightarrow k = \dfrac{1}{20}\ln 2 = \dfrac{\ln 2}{20}$.

25 **a** Using the property of the sum and difference of logarithms, we have:

$$f(x) = \ln x + \ln(x-2) - \ln(x^2-4) = \ln\left(x(x-2)\right) - \ln\left(x^2-4\right) = \ln\frac{x(x-2)}{(x^2-4)}.$$

Applying the formula for the difference of squares, we have: $\ln\dfrac{x(x-2)}{(x-2)(x+2)} = \ln\dfrac{x}{x+2}$.

Note: We can apply the formula for the difference of squares immediately and then apply the formula for the logarithm of a product:

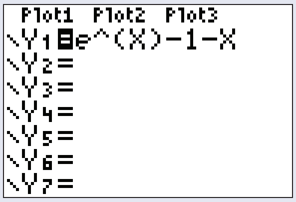
$f(x) = \ln x + \ln(x-2) - \ln\left((x-2)(x+2)\right) = \ln x + \ln(x-2) - \ln(x-2) - \ln(x+2) = \ln x - \ln(x+2)$. Finally, we apply the formula for the difference of logarithms: $f(x) = \ln x - \ln(x+2) = \ln\dfrac{x}{x+2}$.

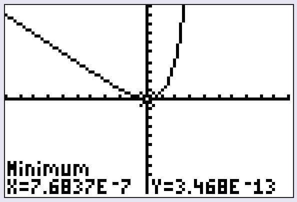
b Switching the variables, $x = \ln\dfrac{y}{y+2}$, we have:

$$e^x = \frac{y}{y+2} \Rightarrow e^x(y+2) = y \Rightarrow ye^x - y = -2e^x \Rightarrow y\left(e^x - 1\right) = -2e^x \Rightarrow y = \frac{-2e^x}{e^x-1}.$$ Hence, $f^{-1}(x) = \dfrac{-2e^x}{e^x-1}$.

Note: If we transfer the y-values to the other side, or if we multiply both the numerator and denominator in the result by -1, the formula will be: $f^{-1}(x) = \dfrac{2e^x}{1-e^x}$.

26 a i





From the data, we can conclude that the minimum value of $f(x)$ is 0.

ii From part **i**, we can conclude that $f(x) \geqslant 0$ for all x; hence, $e^x - 1 - x \geqslant 0 \Rightarrow e^x \geqslant 1 + x$ for all x.

b Let $P(n)$ be the proposition: $(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right) = n+1$.

Basis step:

$P(1)$ is true since $(1 + 1) = 2$ and, for $n = 1$, $n + 1 = 2$.

Inductive step:

Assume $P(k)$ is true for some integer $k \geqslant 1$, that is: $(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right) = k+1$.

Then: $(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right) = (k+1)\left(1+\frac{1}{k+1}\right) = (k+1)\frac{k+1+1}{k+1} = (k+1)+1$

Thus, using the fact that $P(k)$ is true, we establish that $P(k + 1)$ is true, and so $P(n)$ is true for all integers $n \geqslant 1$.

c First transform: $e^{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}} = e^1 e^{\frac{1}{2}} e^{\frac{1}{3}} \ldots e^{\frac{1}{n}}$

Now we will apply **a ii**:

$e^{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}} = e^1 e^{\frac{1}{2}} e^{\frac{1}{3}} \ldots e^{\frac{1}{n}} \geqslant (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)$

And then **b**:

$e^{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}} = e^1 e^{\frac{1}{2}} e^{\frac{1}{3}} \ldots e^{\frac{1}{n}} \geqslant (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right) = n+1 > n$

d $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} > 100 \Rightarrow e^{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}} > e^{100}$

Using the result from part **c**, we can see that any integer greater than e^{100} is a solution.

Chapter 6

Practice questions

1 $\det(A) = \begin{vmatrix} 2x & 3 \\ -4x & x \end{vmatrix} = 2x \cdot x - 3(-4x) = 2x^2 + 12x$

$\det(A) = 14 \Rightarrow 2x^2 + 12x = 14 \Rightarrow x^2 + 6x - 7 = 0 \Rightarrow x_1 = -7, x_2 = 1$

So, $x = -7$ or $x = 1$.

2 **a** $M^2 = \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix}^2 = \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} a^2+4 & 2a-2 \\ 2a-2 & 5 \end{pmatrix}$

b $M^2 = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} a^2+4 & 2a-2 \\ 2a-2 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} \Rightarrow \begin{cases} a^2+4=5 \\ 2a-2=-4 \end{cases}$

The solution $a = -1$ satisfies both equations.

For $a = -1$, we have: $\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$

$M^{-1} = \dfrac{1}{\det(m)}\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} = -\dfrac{1}{3}\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$

The system of equations can be written as:

$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1}\begin{pmatrix} -3 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = -\dfrac{1}{3}\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ 3 \end{pmatrix} = -\dfrac{1}{3}\begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Therefore, the solution is $x = 1, y = -1$.

3 $BA = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}$ and $A = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix}$

$A^{-1} = -\dfrac{1}{4}\begin{pmatrix} 0 & -2 \\ -2 & 5 \end{pmatrix}$

$B = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}A^{-1} = -\dfrac{1}{4}\begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 5 \end{pmatrix} = -\dfrac{1}{4}\begin{pmatrix} -4 & -12 \\ -16 & -48 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$

4 $AX + X = B \Rightarrow \begin{pmatrix} 3 & 1 \\ -5 & 6 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 0 & -3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3a+c & 3b+d \\ -5a+6c & -5b+6d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 0 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 4a+c & 4b+d \\ -5a+7c & -5b+7d \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 0 & -3 \end{pmatrix}$

By comparing the corresponding elements, we obtain two systems of linear equations:

$4a + c = 4$
$\underline{-5a + 7c = 0}$
$a = \dfrac{28}{33}, c = \dfrac{20}{33}$

$4b + d = 8$
$\underline{-5b + 7d = -3}$
$b = \dfrac{59}{33}, d = \dfrac{28}{33}$

So, $a = \dfrac{28}{33}, b = \dfrac{59}{33}, c = \dfrac{20}{33}$, and $d = \dfrac{28}{33}$.

5 $A = \begin{pmatrix} 5 & -2 \\ 7 & 1 \end{pmatrix}$

a $A^{-1} = \dfrac{1}{5+14}\begin{pmatrix} 1 & 2 \\ -7 & 5 \end{pmatrix} = \dfrac{1}{19}\begin{pmatrix} 1 & 2 \\ -7 & 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{19} & \frac{2}{19} \\ \frac{-7}{19} & \frac{5}{19} \end{pmatrix}$

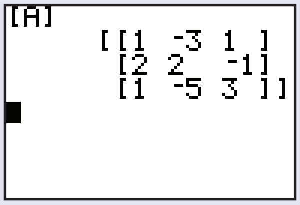
b **i** $XA + B = C \Rightarrow XA = C - B \Rightarrow X = (C - B)A^{-1}$

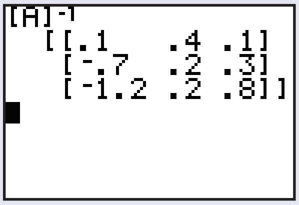
ii $X = \left(\begin{pmatrix} -5 & 0 \\ -8 & 7 \end{pmatrix} - \begin{pmatrix} 6 & 7 \\ 5 & -2 \end{pmatrix}\right)\begin{pmatrix} \frac{1}{19} & \frac{2}{19} \\ \frac{-7}{19} & \frac{5}{19} \end{pmatrix} = \frac{1}{19}\begin{pmatrix} -11 & -7 \\ -13 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -7 & 5 \end{pmatrix} = \frac{1}{19}\begin{pmatrix} 38 & -57 \\ -76 & 19 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$

6 **a** $A + B = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ d & c \end{pmatrix} = \begin{pmatrix} a+1 & b+2 \\ c+d & c+1 \end{pmatrix}$

b $AB = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ d & c \end{pmatrix} = \begin{pmatrix} a+bd & 2a+bc \\ c+d & 3c \end{pmatrix}$

7 **a** Using a GDC:

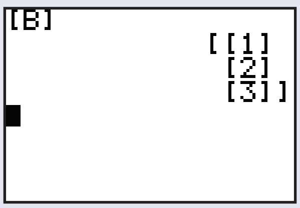




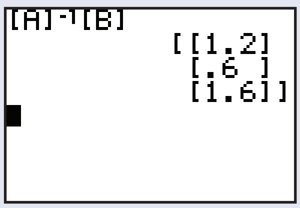
b The system of equations can be written as:

$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

For A as above and



we have:



The solution is: $x = \frac{6}{5}, y = \frac{3}{5}, z = \frac{8}{5}$.

8 Given $C = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix}, D = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$, and $3Q = 2C - D$, we have:

a $Q = \frac{1}{3}(2C - D) = \frac{1}{3}\left[2\begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix} - \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix} -9 & 6 \\ 3 & 14-a \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 1 & \frac{14-a}{3} \end{pmatrix}$

b $CD = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix} = \begin{pmatrix} -10-4 & -4+4a \\ 5-7 & 2+7a \end{pmatrix} = \begin{pmatrix} -14 & 4a-4 \\ -2 & 7a+2 \end{pmatrix}$

c $D^{-1} = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix} = \frac{1}{5a+2}\begin{pmatrix} a & -2 \\ 1 & 5 \end{pmatrix}$

9 **a** If A is the inverse of matrix B, then $AB = BA = I$ must be satisfied. So, we have:

$AB = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix} = \begin{pmatrix} a-6 & 2a-4b-6 & -2a+14 \\ 0 & 5b-9 & 0 \\ 0 & 3b-6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$BA = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}\begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix} = \begin{pmatrix} a-6 & 0 & 0 \\ 3a-8b-5 & 5b-9 & 7b-14 \\ -a+7 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

By comparing the corresponding elements, we can see that the solution $a = 7, b = 2$ satisfies all the equations.

b The system can be written as:

$$B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -2 \\ 3 & 2 & 1 \\ -1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & -2 \\ 3 & 2 & 1 \\ -1 & 1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}$$

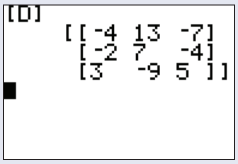
Since $AB = BA = I \Rightarrow B^{-1} = A$, then:

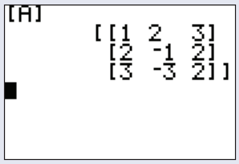
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & -2 \\ 3 & 2 & 1 \\ -1 & 1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

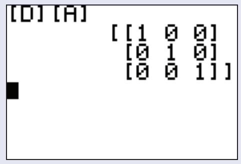
The solution is: $x = -1, y = 2, z = -1$.

10 a $AB = C \Rightarrow B = A^{-1}C$.

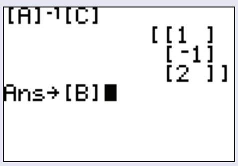
b i For:







ii Matrix $B = A^{-1}C$.



c The coordinates of the point of intersection of the planes are given as the solution of the system of equations that can be written as:

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}C = B$$

The point has coordinates $(1, -1, 2)$.

11 a $\det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} = 1(10-1) - 1(5-2) + 2(1-4) = 0$

b We transform the augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & 2 & 1 & 4 \\ 2 & 1 & 5 & \lambda \end{array}\right) \begin{cases} R_2 - R_1 \\ R_3 - 2R_1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & \lambda - 6 \end{array}\right) \{R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & \lambda - 5 \end{array}\right)$$

The general solution of the system exists if $\lambda - 5 = 0 \Rightarrow \lambda = 5$.

For $\lambda = 5$ we have:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)\{R_1 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

For $z = t \Rightarrow x = 2 - 3t,\ y = 1 + t$.

12 If $X^3 = 0$ we have:

$$(I - X)(I + X + X^2) = II + IX + IX^2 - XI - XX - XX^2 = I + X + X^2 - X - X^2 - X^3 = I - X^3 = I - 0 = I$$

$$(I + X + X^2)(I - X) = II - IX + XI - XX + X^2I - X^2X = I - X + X - X^2 + X^2 - X^3 = I - X^3 = I - 0 = I$$

We can conclude: $(I - X)^{-1} = (I + X + X^2)$ and $(I + X + X^2)^{-1} = (I - X)$.

Chapter 7

Practice questions

Solution Paper 1 type

1 **a** The length is a function of time, so we are looking for the value of the function when $t = 2$:

$$L(2) = 110 + 25\cos(2\pi \cdot 2) = 110 + 25\cos(4\pi) = 110 + 25 \cdot 1 = 135 \text{ cm}$$

b We are looking for the minimum value of a function of the form $y = a\cos[b(x + c)] + d$. Since the smallest value of cosine is -1, the minimum value of L will be:

$$L_{\min} = 110 + 25 \cdot (-1) = 85 \text{ cm}$$

Note: We have found the minimum value of the function using the fact that the minimum value of cosine is -1. We could have found the result differently: The minimum value of a function of the form $y = a\cos[b(x + c)] + d$ is equal to the vertical shift d minus the amplitude $|a|$; so we have: $\min = d - |a| = 110 - 25 = 85$.

c We have to find the least value of t such that $L = 85$. From part **b** we can see that the lowest value will be obtained when the cosine equals -1; hence, $2\pi t = \pi \Rightarrow t = \frac{1}{2}$.

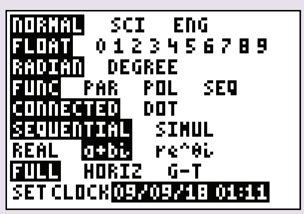
Note: We could find t by solving the equation: $110 + 25\cos(2\pi t) = 85 \Rightarrow 25\cos(2\pi t) = 85 - 110 \Rightarrow \cos(2\pi t) = -1$.

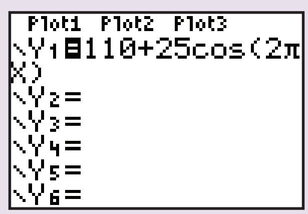
$$2\pi t = \pi \Rightarrow t = \frac{1}{2} \text{ sec}$$

d We have to determine the period of a function of the form $y = a\cos[b(x + c)] + d$. So, the period is $\frac{2\pi}{|b|} = \frac{2\pi}{2\pi} = 1$ sec.

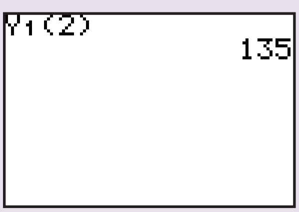
Solution Paper 2 type

1 We will observe the function $L = 110 + 25\cos(2\pi t)$.
Firstly, we have to select radian measure and then input the function.





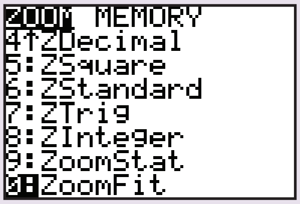
a We have to find L(2):

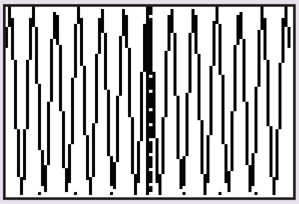


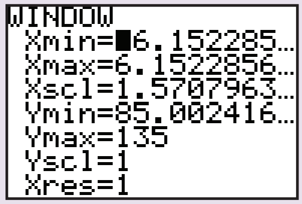
b–c For the remaining parts of the question, we will use the graph of the function. So, we have to set a suitable window.

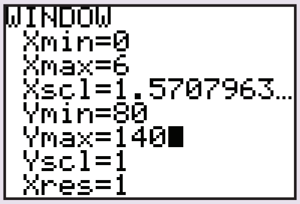
The minimum value is the vertical shift minus the amplitude, and the maximum value is the vertical shift plus the amplitude. Therefore, we can use y-values from (less than) $110 - 25 = 85$ to (greater than) $110 + 25 = 135$.

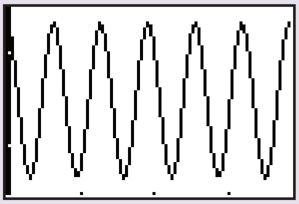
Another way would be to use the ZoomFit feature. The window will not be suitable for all the calculations, but we can read out the range and then just extend it a bit in both directions.



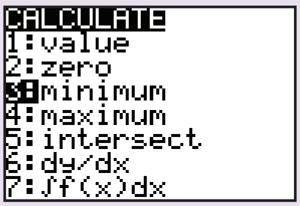


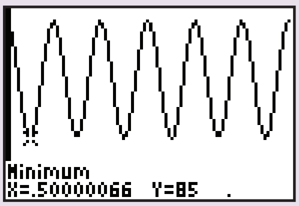




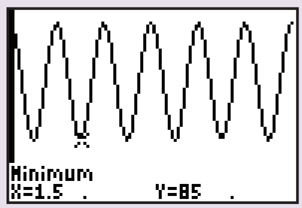


We can solve both parts **b** and **c** by finding the minimum value with the smallest x.





d We can find the period of a trigonometric function by finding two successive minimum points, or two successive maximum points, and then subtracting their x-coordinates. We already know the smallest positive minimum has $x = 0.5$, so we have to find the next positive minimum.



Hence, we can conclude that the period is $1.5 - 0.5 = 1$.

2 $2\sin^2 x - \cos x + 1 = 0$

$2\left(1 - \cos^2 x\right) - \cos x + 1 = 0$ Use the Pythagorean identity $\sin^2 x = 1 - \cos^2 x$.

$-2\cos^2 x - \cos x + 3 = 0$

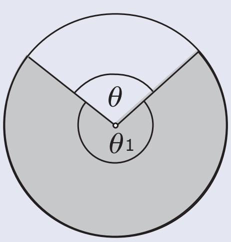
$-2t^2 - t + 3 = 0 \Rightarrow t_1 = -1.5, t_2 = 1$ Substitute $\cos x = t$.

$\cos x = 1.5 \Rightarrow$ has no solution

$\cos x = 1 \Rightarrow x = 0, 2\pi$

Solutions are: $0, 2\pi$.

3



The perimeter of the shaded sector contains two radii and an arc. So, the length of the arc can be calculated: $25 = 2 \cdot 6 + s \Rightarrow s = 13$. We can use the arc length formula to find the angle θ_1 in radians:

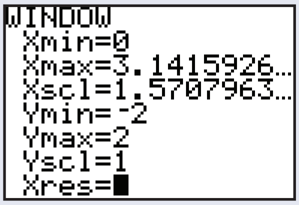
$$s = r\theta_1 \Rightarrow 13 = 6 \cdot \theta_1 \Rightarrow \theta_1 = \frac{13}{6}$$

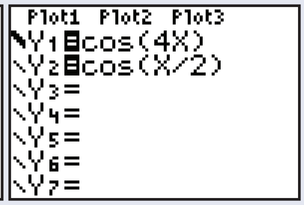
For angles θ and θ_1:

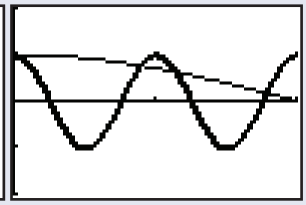
$$\theta + \theta_1 = 2\pi \Rightarrow \theta = 2\pi - \frac{13}{6} \approx 4.12$$

4 **a** **i** The amplitude of function f is 1, so the minimum value of the function is -1.

ii Period of $g = \dfrac{2\pi}{\frac{1}{2}} = 4\pi$.







b It is quickest to find the number of solutions from the graph.

We can see that there are four points of intersection; hence, there are four solutions to the equation.

5 For $d = p + q\cos\left(\dfrac{2\pi}{m}t\right)$:

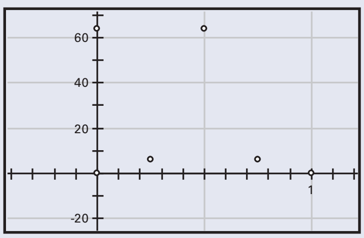
a To find p we have to determine the mid-line; therefore, we have to find the average of the function's maximum and minimum value:

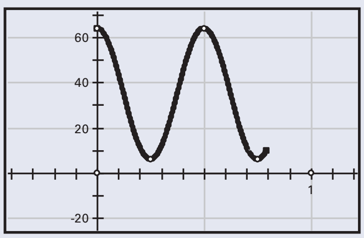
$$p = \frac{64 + 6}{2} = 35$$

b To determine q, we have to determine the amplitude. The amplitude is the difference between the function's maximum value and the mid-line: $|q| = 64 - 35 = 29$. So, q is 29 or -29. From the given data, we can establish that the graph starts from the maximum value, so q is positive; hence, $q = 29$.

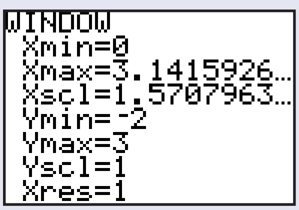
c From the data, we can see that the distance between two successive maximum points (or minimum points) is 0.5 seconds. So, the period is 0.5.
Using the formula for period, we have: period $= \dfrac{2\pi}{\frac{2\pi}{m}} = m$. Hence, $m = 0.5$.

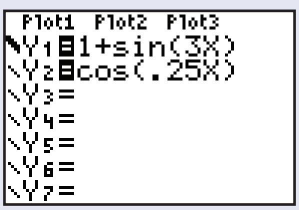
Note: It is useful to highlight the basic data using a rough sketch.

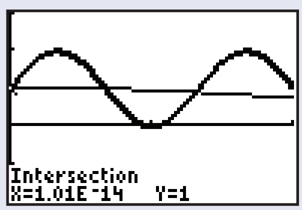


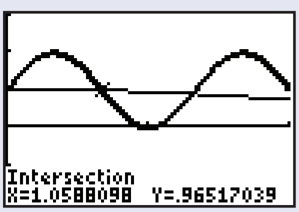


6









Solutions are: 0, 1.06, 2.05.

Note: The question does not tell us to use a specific method to solve the equation, so we can choose any method. The most suitable method is graphical, because we can see the number of solutions and find them all.

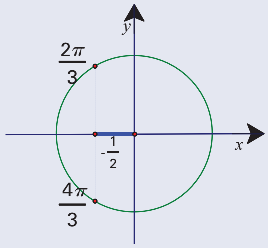
7 **a** Use the substitution $\cos x = t$:

$2t^2 + 5t + 2 = 0 \Rightarrow t_1 = -2, t_2 = -\frac{1}{2}$

$\cos x = -2$ has no solutions

$\cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Solutions are: $\frac{2\pi}{3}, \frac{4\pi}{3}$.



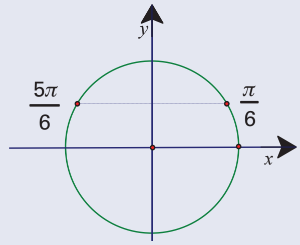
b Use double angle identity for sine:

$2 \sin x \cos x - \cos x = 0$

Factorize:

$$\cos x(2\sin x - 1) = 0 \Rightarrow \begin{cases} \cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \\ 2\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6} \end{cases}$$

Solutions are: $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.



8 Given $\frac{\pi}{2} < x < \pi$, it follows that $\sin x > 0$ and $\cos x < 0$.

a Using the Pythagorean identity for sine, we have: $\sin^2 x = 1 - \cos^2 x = 1 - \frac{8}{9} = \frac{1}{9}$.

Using the fact that $\sin x > 0$, we have: $\sin x = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

b Using the double angle identity for cosine, we have: $\cos 2x = 2\cos^2 x - 1 = 2 \cdot \frac{8}{9} - 1 = \frac{16}{9} - 1 = \frac{7}{9}$.

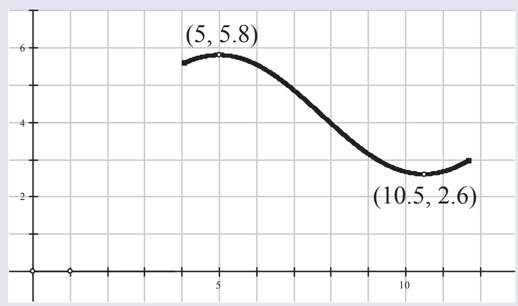
c Using $\cos x < 0$, we have: $\cos x = -\sqrt{\frac{8}{9}} = -\frac{\sqrt{8}}{3} = -\frac{2\sqrt{2}}{3}$. Hence,

$\sin 2x = 2 \sin x \cos x = 2 \cdot \frac{1}{3} \cdot \left(-\frac{2\sqrt{2}}{3}\right) = -\frac{4\sqrt{2}}{9}$.

9 We have to interpret the data as points on a graph:

High (maximum tide) at 5:00 with depth of 5.8 m $\rightarrow$ (5, 5.8)

Low (minimum) tide at 10:30 with depth 2.6 m $\rightarrow$ (10.5, 2.6)

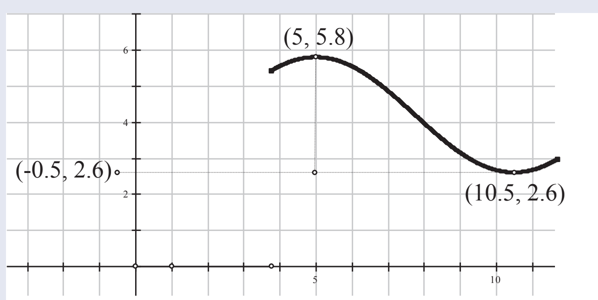


a The function can be written in the form:
$d = A \sin(B(x + C)) + D$ (or similar with the cosine function). Firstly, we will determine D, so we have to find the average of the function's maximum and minimum value:

$D = \frac{5.8 + 2.6}{2} = 4.2$

Next we will determine A. The amplitude of the function is the difference between the function's maximum value and the mid-line:

$|A| = 5.8 - 4.2 = 1.6$. So, A is 1.6 or -1.6.



From the data, we can see that the earlier minimum is reached at $(-0.5, 2.6)$, so the graph after the mid-line first reaches the maximum value, then the minimum value; therefore, A is positive, $A = 1.6$.

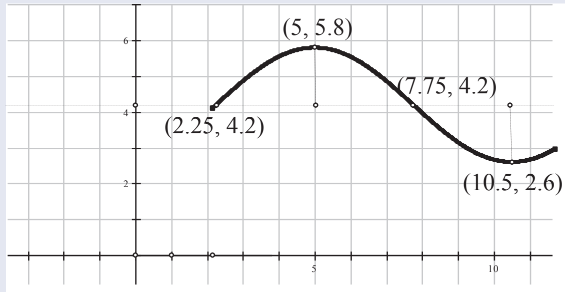
We can also see from the data that the distance between successive maximum and minimum points is 5.5 hours. So, half of the period = 5.5.

Using the formula for period, we have: period $= \frac{2\pi}{B} \Rightarrow 11 = \frac{2\pi}{B} \Rightarrow B = \frac{2\pi}{11}$.

To determine the horizontal translation (phase shift), we have to find the points of the graph on the mid-line.

Abscissae of those points are on a distance of a $\frac{1}{4}$ of the period from the abscissa of the maximum point. Hence,

$5 \pm \frac{1}{4} \cdot 11$, so $\frac{9}{4}$ and $\frac{31}{4}$.

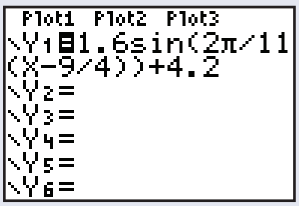


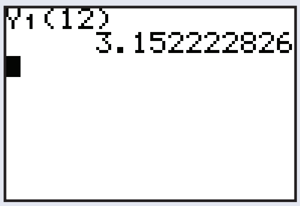
Thus, $C = -\frac{9}{4}$ and the function is:

$$d = 1.6 \sin\left(\frac{2\pi}{11}\left(x - \frac{9}{4}\right)\right) + 4.2.$$

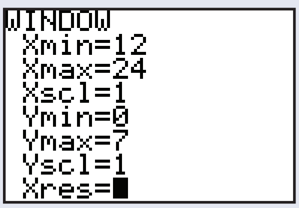
b We have to find the value of the function at 12 noon, that is, 12 hours after midnight; therefore,

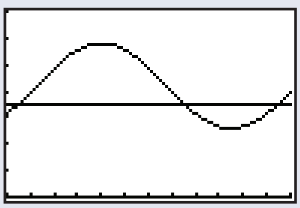
$$d(12) = 1.6 \sin\left(\frac{2\pi}{11}\left(12 - \frac{9}{4}\right)\right) + 4.2 \approx 3.15 \text{ metres.}$$

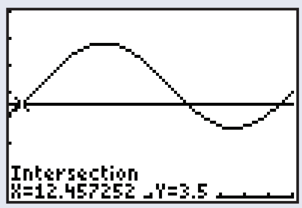


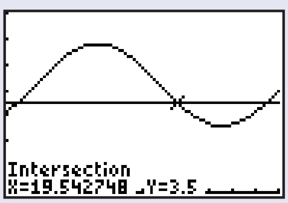


c We have to find the time interval in which the value of d is greater than 3.5.

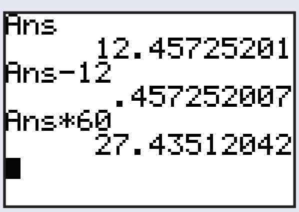


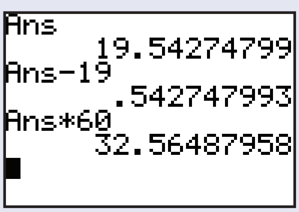






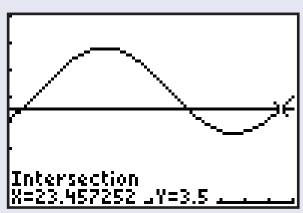
We have to convert from decimal numbers to minutes:





Therefore, from about 12:27 pm to 7:33 pm the boat can dock safely.

We can also see that around midnight the boat can dock safely again.

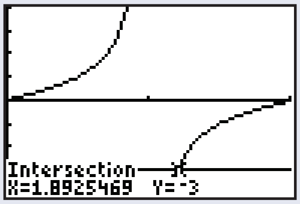


10 Use the substitution $\tan x = t$:

$t^2 + 2t - 3 = 0 \Rightarrow t_1 = 1, t_2 = -3$

$\tan x = 1 \Rightarrow x = \frac{\pi}{4}$

$\tan x = -3$ cannot be solved exactly

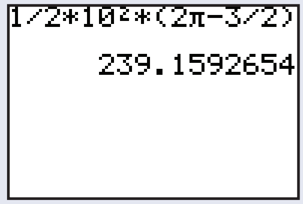


Solutions are: $\frac{\pi}{4}$, 1.89.

11 a We will use the arc length formula: $s = r\theta = 10 \cdot \frac{3}{2} = 15$ cm

b The angle in the shaded region is $2\pi - \frac{3}{2}$; therefore, using the area of a sector formula:

$A = \frac{1}{2} r^2 \theta = \frac{1}{2} \cdot 10^2 \left(2\pi - \frac{3}{2}\right) \approx 239\ \text{cm}^2$



12 The function $f(x) = \frac{5}{2}\cos\left(2x - \frac{\pi}{2}\right)$ reaches a maximum value of $\frac{5}{2}$ and a minimum value of $-\frac{5}{2}$. That means that $-\frac{5}{2} \leqslant f(x) \leqslant \frac{5}{2}$. The function will not reach values greater than $\frac{5}{2}$ or less than $-\frac{5}{2}$. Hence, for $k > \frac{5}{2}$ and $k < -\frac{5}{2}$, the equation will have no solutions.

13 From the graph, we can see that the point (0, 1) is on the mid-line, so $k = 1$.
To determine a, we need to find the amplitude. The amplitude is the difference between the function's maximum value and the mid-line: $|a| = 3 - 1 = 2$; so, a is 2 or -2. We can see that the graph, starting from initial position at $x = 0$, first reaches the minimum value, and then the maximum value, so a is negative; hence, $a = -2$.

14 We can write $\tan^2 \alpha$ using secant only: $\tan^2 \alpha = \frac{\sin^2 \alpha}{\cos^2 \alpha} = \frac{1 - \cos^2 \alpha}{\cos^2 \alpha} = \frac{1}{\cos^2 \alpha} - 1 = \sec^2 \alpha - 1$

So, the equation is: $2\sec^2 \alpha - 2 - 5\sec \alpha - 10 = 0 \Rightarrow 2\sec^2 \alpha - 5\sec \alpha - 12 = 0$

Using the substitution $\sec \alpha = t$, we have: $2t^2 - 5t - 12 = 0 \Rightarrow t_1 = -\frac{3}{2}, t_2 = 4$. Since α is in the second quadrant, secant should be negative and the solution is $\sec \alpha = -\frac{3}{2}$.

15 a Using the compound angle formula for sine, we have: $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

From the upper triangle, we can see that: $\sin \alpha = \frac{15}{17}$, $\cos \alpha = \frac{8}{17}$

From the lower triangle, we can see that: $\sin \beta = \frac{6}{10}$, $\cos \beta = \frac{8}{10}$

So, $\sin(\alpha + \beta) = \frac{15}{17} \cdot \frac{8}{10} + \frac{8}{17} \cdot \frac{6}{10} = \frac{168}{170} = \frac{84}{85}$.

b $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \frac{8}{17}\frac{8}{10} - \frac{15}{17}\frac{6}{10} = \frac{-26}{170} = -\frac{13}{85}$

c $\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\frac{84}{85}}{-\frac{13}{85}} = -\frac{84}{13}$

Note: We can find $\tan \alpha$ and $\tan \beta$ from the drawing, and then use the compound formula for tangent.

16 From the diagram, we can see that: $\sin p^\circ = \frac{1}{\sqrt{1^2 + 2^2}} = \frac{1}{\sqrt{5}}$ and $\cos p^\circ = \frac{2}{\sqrt{1^2 + 2^2}} = \frac{2}{\sqrt{5}}$.

Hence: $\sin 2p^\circ = 2\sin p^\circ \sin p^\circ = 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{4}{5}$.

For $\sin 3p^\circ$, we can use the compound angle formula: $\sin 3p^\circ = \sin\left(2p^\circ + p^\circ\right) = \sin 2p^\circ \cos p^\circ + \cos 2p^\circ \sin p^\circ$

So, we have to find $\cos 2p^\circ = \cos^2 p^\circ - \sin^2 p^\circ = \frac{4}{5} - \frac{1}{5} = \frac{3}{5}$. Finally, we have:

$$\sin 3p^\circ = \frac{4}{5}\frac{2}{\sqrt{5}} + \frac{3}{5}\frac{1}{\sqrt{5}} = \frac{11}{5\sqrt{5}} = \frac{11\sqrt{5}}{25}.$$

17 If B is obtuse, then the sine is positive and cosine negative.

a $\sin B = \dfrac{5}{\sqrt{5^2+12^2}} = \dfrac{5}{13}$

b $\cos B = -\dfrac{12}{\sqrt{5^2+12^2}} = -\dfrac{12}{13}$

c $\sin 2B = 2\sin B\cos B = 2\dfrac{5}{13}\left(-\dfrac{12}{13}\right) = -\dfrac{120}{169}$

d $\cos 2B = \cos^2 B - \sin^2 B = \left(-\dfrac{12}{13}\right)^2 - \left(\dfrac{5}{13}\right)^2 = \dfrac{119}{169}$

Note: In the solution, we have used the property of angles and sides in a right triangle.

We can find the solution by using the Pythagorean identity for sine and cosine:

$$\frac{\sin x}{\cos x} = -\frac{5}{12} \Rightarrow \sin x = -\frac{5}{12}\cos x$$

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \left(-\frac{5}{12}\cos x\right)^2 + \cos^2 x = 1 \Rightarrow \frac{169}{144}\cos^2 x = 1 \Rightarrow \cos x = -\sqrt{\frac{144}{169}} = -\frac{12}{13}$$

Finally, $\sin x = -\dfrac{5}{12}\left(-\dfrac{12}{13}\right) = \dfrac{5}{13}$. Now we can continue using the double angle formulae.

18 Using the double angle formula for tangent, we have: $\tan 2\theta = \dfrac{3}{4} \Rightarrow \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{3}{4} \Rightarrow 8\tan\theta = 3 - 3\tan^2\theta$

Using the substitution $\tan\theta = t$, we will have a quadratic equation:

$3t^2 + 8t - 3 = 0 \Rightarrow t_1 = -3, t_2 = \dfrac{1}{3}$. Hence, the possible values of $\tan\theta$ are: $-3, \dfrac{1}{3}$.

19 We will use the compound angle formula for sine:

$$\sin x\cos\alpha - \cos x\sin\alpha = k(\sin x\cos\alpha + \cos x\sin\alpha)$$

Then we will divide both sides by $\cos x\cos\alpha$:

$$\frac{\sin x\cos\alpha}{\cos x\cos\alpha} - \frac{\cos x\sin\alpha}{\cos x\cos\alpha} = k\left(\frac{\sin x\cos\alpha}{\cos x\cos\alpha} + \frac{\cos x\sin\alpha}{\cos x\cos\alpha}\right) \Rightarrow$$

$$\tan x - \tan\alpha = k(\tan x + \tan\alpha) \Rightarrow \tan x - k\tan x = k\tan\alpha + \tan\alpha \Rightarrow \tan x(1-k) = \tan\alpha(k+1) \Rightarrow$$

$$\tan x = \frac{\tan\alpha(k+1)}{1-k}$$

Note: If we rearrange the equation differently, we obtain the result $\tan x = \dfrac{-(k+1)}{k-1}\tan\alpha$, which is equivalent to the above result.

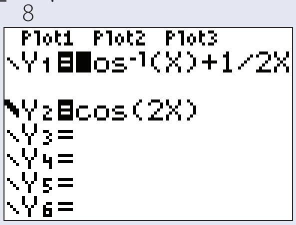
20 We can see that either $\tan 2\theta = 1$ or $\tan 2\theta = -1$.

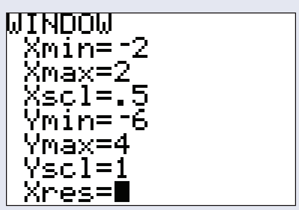
So, $\tan 2\theta = 1 \Rightarrow 2\theta = \dfrac{\pi}{4} + k\pi \Rightarrow \theta = \dfrac{\pi}{8} + k\dfrac{\pi}{2} \Rightarrow \theta = \dfrac{\pi}{8}, \dfrac{\pi}{8} + \dfrac{\pi}{2} = -\dfrac{3\pi}{8}$

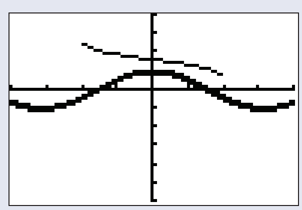
$$\tan 2\theta = -1 \Rightarrow 2\theta = -\frac{\pi}{4} + k\pi \Rightarrow \theta = -\frac{\pi}{8} + k\frac{\pi}{2} \Rightarrow \theta = -\frac{\pi}{8}, -\frac{\pi}{8} + \frac{\pi}{2} = \frac{3\pi}{8}$$

Hence, the solutions are: $\pm\dfrac{\pi}{8}, \pm\dfrac{3\pi}{8}$.

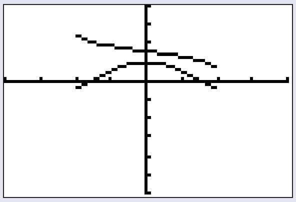
21 a We have to set the window on the GDC to be the same as on the grid.







The domain of the functions is $-1 \leqslant x \leqslant 1$, so we just have to draw this part of the graph.



b No solutions

c Cosine function has a range from -1. But, here, its domain is restricted. Since cosine is an even function, it is enough to observe its behaviour on the interval from 0 to 1. Hence, the function is decreasing on this interval; its largest value is $\cos 0 = 1$, and the smallest value $\cos 2$. Hence, the range is the set $[\cos 2, 1]$.

22 Let $C\hat{A}D = \theta$ and $AC = x$.

From the drawing, we can see that: $\tan\theta = \frac{2}{x} \Rightarrow x = \frac{2}{\tan\theta}$ and $\tan 2\theta = \frac{5}{x} \Rightarrow x = \frac{5}{\tan 2\theta}$.

Hence: $\frac{2}{\tan\theta} = \frac{5}{\tan 2\theta}$

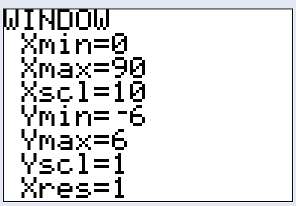
Using the double angle formula for tangent, we have: $\frac{2}{\tan\theta} = \frac{5}{\frac{2\tan\theta}{1-\tan^2\theta}} \Rightarrow 2 = \frac{5(1-\tan^2\theta)}{2} \Rightarrow 4 = 5 - 5\tan^2\theta \Rightarrow \tan^2\theta = \frac{1}{5}$

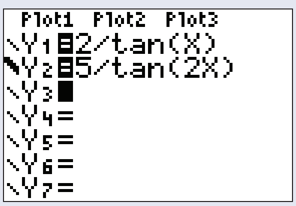
Since θ is an angle in a right triangle, its tangent has to be positive.

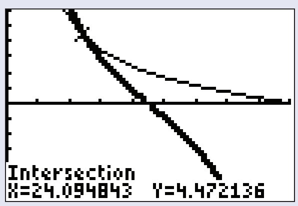
So, $\tan\theta = \frac{1}{\sqrt{5}} \Rightarrow \theta \approx 24.1°$.

Note: We can solve the equation $\frac{2}{\tan\theta} = \frac{5}{\tan 2\theta}$ graphically.

We are looking for the angle when those two sides are the same.







23 We have to solve the equation $16\sec\left(\frac{\pi x}{36}\right) - 32 = -16 + 10$. Hence,

$$\sec\left(\frac{\pi x}{36}\right) = \frac{26}{16} = \frac{13}{8} \Rightarrow \frac{1}{\cos\left(\frac{\pi x}{36}\right)} = \frac{13}{8} \Rightarrow \cos\left(\frac{\pi x}{36}\right) = \frac{8}{13}$$

We need the first positive solution, so:

$$\frac{\pi x}{36} = \arccos\left(\frac{8}{13}\right) \Rightarrow x = \frac{36}{\pi}\arccos\left(\frac{8}{13}\right)$$

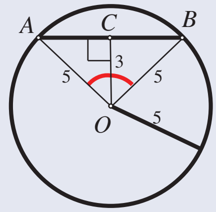
Hence, the solution of the equation is: $\frac{36}{\pi}\arccos\left(\frac{8}{13}\right)$, and the width of the water surface in the channel is:

$$2 \cdot \frac{36}{\pi}\arccos\left(\frac{8}{13}\right) = \frac{72}{\pi}\arccos\left(\frac{8}{13}\right) \text{ cm.}$$

Chapter 8

Practice questions

1 The shortest distance from AB to O is the perpendicular from O to AB, so $OC = 3$ units, and $CB = \sqrt{5^2 - 3^2} = 4$ units. Therefore, $AB = 2 \cdot 4 = 8$ units.



We are asked to find the exact value of the sine of the angle, which means that we have to avoid all calculator use!

Method I:

In triangle AOB all sides are given, so we can find angle $A\hat{O}B$ using the law of cosines:

$$\cos A\hat{O}B = \frac{5^2 + 5^2 - 8^2}{2 \cdot 5 \cdot 5} = -\frac{7}{25}$$

Using the Pythagorean identity for sine and cosine, and taking into account the fact that the sine of an angle in a triangle is always positive, we have:

$$\sin A\hat{O}B = \sqrt{1 - \left(-\frac{7}{25}\right)^2} = \sqrt{\frac{25^2 - 7^2}{25^2}} = \frac{\sqrt{18 \cdot 32}}{25} = \frac{\sqrt{9 \cdot 2 \cdot 2 \cdot 16}}{25} = \frac{3 \cdot 2 \cdot 4}{25} = \frac{24}{25}$$

Method II:

From the right triangle OBC, we can find acute angle $O\hat{B}C$: $\sin O\hat{B}C = \frac{3}{5}$

Using the law of sines in triangle AOB, we have: $\frac{\sin A\hat{O}B}{8} = \frac{\sin O\hat{B}C}{5} \Rightarrow \sin A\hat{O}B = \frac{8 \cdot \sin O\hat{B}C}{5} = \frac{8\frac{3}{5}}{5} = \frac{24}{25}$

Method III:

From the right triangle OCB, we can find the sine and cosine of the acute angle $O\hat{B}C$:

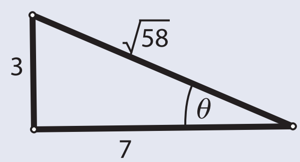
$$\sin O\hat{B}C = \frac{4}{5} \text{ and } \cos O\hat{B}C = \frac{3}{5}$$

The angle $A\hat{O}B = 2 \cdot O\hat{B}C$, so, using the double angle identity for sine, we have:

$$\sin A\hat{O}B = 2 \sin O\hat{B}C \cdot \cos O\hat{B}C = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$$

2 In the given right triangle, $\tan \theta = \frac{3}{7}$. Therefore, the triangle has legs of 3 and 7 and hypotenuse of $\sqrt{3^2 + 7^2} = \sqrt{58}$.

Hence, $\sin \theta = \frac{3}{\sqrt{58}}$ and $\cos \theta = \frac{7}{\sqrt{58}}$.



Using double angle identities, we have:

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{3}{\sqrt{58}} \cdot \frac{7}{\sqrt{58}} = \frac{42}{58} = \frac{21}{29}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \frac{49}{58} - \frac{9}{58} = \frac{20}{29}$$

3 In the triangle SSS is given, so we can find any angle using the law of cosines. The largest angle is opposite the longest side; therefore, we are looking for the angle opposite the side of size 7.

$$\cos \theta = \frac{4^2 + 5^2 - 7^2}{2 \cdot 4 \cdot 5} = -\frac{1}{5} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{5}\right) \approx 101.5^\circ$$

4 If A is obtuse, then $\cos A$ is negative. Therefore, using the Pythagorean identity for sine and cosine, we have:

$\cos A = -\sqrt{1 - \sin^2 A} = -\sqrt{1 - \frac{25}{169}} = -\frac{12}{13}$. Hence, using the double angle identity for sine, we have:

$$\sin 2A = 2 \sin A \cos A = 2\frac{5}{13}\left(-\frac{12}{13}\right) = -\frac{120}{169}.$$

5 **a** From the right triangle *BQP*, we have: $\tan 36^\circ = \frac{PQ}{40} \Rightarrow PQ = 40 \tan 36^\circ \approx 29.1$ m

b In triangle *ABQ*, the angle $A\hat{Q}B = 180^\circ - 70^\circ - 30^\circ = 80^\circ$. Using the law of sines, we can find side *AB*:

$$\frac{AB}{\sin 80^\circ} = \frac{40}{\sin 70^\circ} \Rightarrow AB = \frac{40 \sin 80^\circ}{\sin 70^\circ} \approx 41.9 \text{ m}$$

6 In triangle *ABC*, SSS is given; hence, we can find angle $C\hat{A}B$ using the law of cosines:

$$\cos C\hat{A}B = \frac{48^2 + 32^2 - 56^2}{2 \cdot 48 \cdot 32} \Rightarrow C\hat{A}B \approx 86.4^\circ$$

```
(48²+32²-56²)/(2
*48*32)
           .0625
cos⁻¹(Ans
      86.4166783
```

7 **a** In the triangle SSS is given, so we can find any angle using the law of cosines. The smallest angle is opposite the shortest side; therefore, we are looking for the angle opposite the side of size 5.

$$\cos \theta = \frac{7^2 + 8^2 - 5^2}{2 \cdot 7 \cdot 8} \Rightarrow \theta = \cos^{-1}\left(\frac{11}{14}\right) \approx 38.2^\circ$$

b Using the area formula with sides 7 and 8 and the included angle from **a**: $A = \frac{1}{2} \cdot 7 \cdot 8 \sin \theta \approx 17.3 \text{ cm}^2$

```
(7²+8²-5²)/(2*7*
8)
    .7857142857
cos⁻¹(Ans
      38.2132107
```

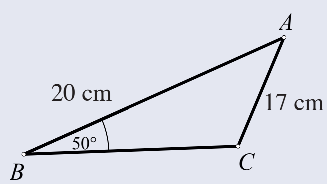
```
8)
    .7857142857
cos⁻¹(Ans
      38.2132107
1/2*7*8*sin(Ans)

     17.32050808
```

8 **a** In Triangle 2, *ABC*, SSA is given.

We can use the law of sines to find angle $A\hat{C}B$:

$$\frac{\sin A\hat{C}B}{20} = \frac{\sin 50^\circ}{17} \Rightarrow \sin A\hat{C}B = \frac{20 \cdot \sin 50^\circ}{17} \approx 0.901\,2287\ldots$$

In Triangle 2, angle $A\hat{C}B$ is obtuse, so $A\hat{C}B = 180^\circ - \sin^{-1} 0.901\,2288 \approx 116^\circ$.



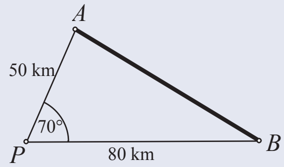

```
20sin(50)/17
     .9012287566
sin⁻¹(Ans
     64.32005482
180-Ans
     115.6799452
```

b In Triangle 1, angle $A\hat{C}B$ is acute, so $A\hat{C}B = \sin^{-1} 0.901\,2288 \approx 64.3^\circ$; therefore, angle $B\hat{A}C = 180^\circ - 50^\circ - B\hat{C}A \approx 65.68^\circ$. Hence, the area is:

$$A = \frac{1}{2} 20 \cdot 17 \cdot \sin B\hat{A}C \approx 155 \text{ cm}^2.$$

9 In 2.5 hours, boat A covers a distance of $20 \cdot 2.5 = 50$ km, and boat B covers a distance of $32 \cdot 2.5 = 80$ km.

In triangle PAB, SAS is given, so we can find side AB using the law of cosines:

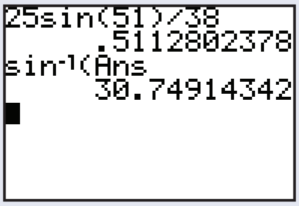


$AB = \sqrt{50^2 + 80^2 - 2 \cdot 50 \cdot 80 \cos 70^\circ} \approx 78.5$ km

Therefore, after 2.5 hours, the boats are approximately 78.5 km apart.

10 In triangle JKL, SSA is given. The angle $K\hat{J}L$ is the angle opposite the longer side, so there will be a unique triangle with the given dimensions. We can determine angle $J\hat{K}L$ using the law of sines:

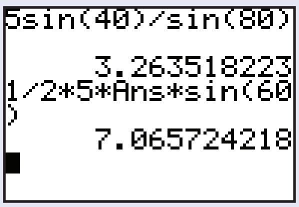
$$\frac{\sin J\hat{K}L}{25} = \frac{\sin 51^\circ}{38} \Rightarrow \sin J\hat{K}L = \frac{25 \sin 51^\circ}{38} \approx 0.51128 \Rightarrow J\hat{K}L \approx 31^\circ$$



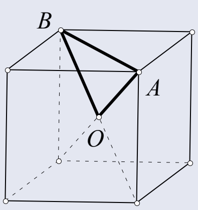
11 a In triangle ABC, ASA is given, so we can find AB using the law of sines. Angle $\hat{A} = 180^\circ - 60^\circ - 40^\circ = 80^\circ$.

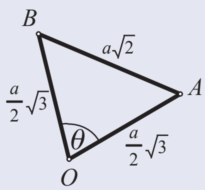
$$\frac{AB}{\sin 40^\circ} = \frac{5}{\sin 80^\circ} \Rightarrow AB = \frac{5 \sin 40^\circ}{\sin 80^\circ} \approx 3.26 \text{ cm}$$

b Area $= \frac{1}{2} 5 \cdot AB \sin 60^\circ \approx 7.07$ cm^2



12





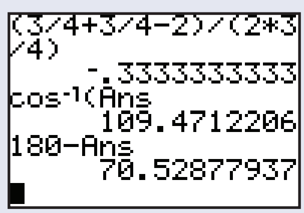
In triangle OAB, sides OA and OB are each half the length of the space diagonals, and side AB is the side diagonal of the cube.

$AB = a\sqrt{2}$

$$OA = OB = \frac{1}{2}\sqrt{\left(a\sqrt{2}\right)^2 + a^2} = \frac{a}{2}\sqrt{3}$$

In triangle OAB, SSS is given. We can use the law of cosines to find angle θ:

$$\cos \theta = \frac{\left(\frac{a}{2}\sqrt{3}\right)^2 + \left(\frac{a}{2}\sqrt{3}\right)^2 - \left(a\sqrt{2}\right)^2}{2\left(\frac{a}{2}\sqrt{3}\right)\left(\frac{a}{2}\sqrt{3}\right)} = \frac{\frac{3a^2}{4} + \frac{3a^2}{4} - 2a^2}{2\,\frac{3a^2}{4}} = -\frac{1}{3} \Rightarrow \theta \approx 109.47^\circ$$



The acute angle between the diagonals is $180^\circ - \theta \approx 70.5^\circ$.

13 a $BC = \sqrt{104^2 + 65^2 - 2 \cdot 104 \cdot 65 \cos 60^\circ} = \sqrt{10\,816 + 4225 - 6760} = \sqrt{8281} = 91$ m

b Area $= \frac{1}{2} 65 \cdot 104 \sin 60^\circ = \frac{1}{2} 65 \cdot 104 \cdot \frac{\sqrt{3}}{2} = 1690\sqrt{3}$

c i Area of $A_1 = \frac{1}{2} 65 \cdot x \sin 30^\circ = \frac{1}{2} 65 \cdot x \cdot \frac{1}{2} = \frac{65x}{4}$

ii Area of $A_2 = \frac{1}{2} 104 \cdot x \sin 30^\circ = \frac{1}{2} 104 \cdot x \cdot \frac{1}{2} = 26x$

iii From **b** we know that the area is $A = 1690\sqrt{3}$. Since $A = A_1 + A_2$, we have:

$1690\sqrt{3} = \frac{65x}{4} + 26x \Rightarrow 1690\sqrt{3} = \frac{169x}{4} \Rightarrow 40\sqrt{3} = x.$

d i The angles $A\hat{D}C + A\hat{D}B = 180^\circ$; hence, $\sin A\hat{D}C = \sin A\hat{D}B$.

ii Method I:

We will use the sine rule in triangle ADB: $\frac{BD}{\sin 30^\circ} = \frac{65}{\sin A\hat{D}B} \Rightarrow BD = \frac{65 \sin 30^\circ}{\sin A\hat{D}B}$

Next, we will use the sine rule in triangle ADC: $\frac{DC}{\sin 30^\circ} = \frac{104}{\sin A\hat{D}C} \Rightarrow DC = \frac{104 \sin 30^\circ}{\sin A\hat{D}C} = \frac{104 \sin 30^\circ}{\sin A\hat{D}B}$

Therefore:

$$\frac{BD}{DC} = \frac{\frac{65 \sin 30^\circ}{\sin A\hat{D}B}}{\frac{104 \sin 30^\circ}{\sin A\hat{D}B}} = \frac{65}{104} = \frac{5}{8}$$

Method II:

We can find areas $A_1 = \frac{65x}{4}$ and $A_2 = 26x$ using the sides BD and DC and angles $A\hat{D}C$ and $A\hat{D}B$:

$$\frac{\frac{65x}{4}}{26x} = \frac{A_1}{A_2} = \frac{\frac{1}{2} xBD \sin A\hat{D}B}{\frac{1}{2} xDC \sin A\hat{D}C} = \frac{BD}{DC}$$

Since $\frac{\frac{65x}{4}}{26x} = \frac{5}{8}$, we have established that $\frac{BD}{DC} = \frac{5}{8}$.

14 a Using the law of sines, we have:

$\frac{x}{\sin 45^\circ} = \frac{x-2}{\sin 30^\circ} \Rightarrow \sqrt{2}x = 2x - 4 \Rightarrow x\left(2 - \sqrt{2}\right) = 4 \Rightarrow x = \frac{4}{2-\sqrt{2}} = \frac{4\left(2+\sqrt{2}\right)}{4-2} = 4 + 2\sqrt{2}$

b $A = \frac{1}{2} x(x-2) \sin 105^\circ$. So, we have to determine the exact value of $\sin 105^\circ$. Using the compound formula for sine, we have: $\sin 105^\circ = \sin\left(60^\circ + 45^\circ\right) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ = \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

Now we have: $A = \frac{1}{2}\left(4 + 2\sqrt{2}\right)(2 + 2\sqrt{2})\frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\left(2+\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(\sqrt{6}+\sqrt{2}\right)}{2}$

$A = \frac{\left(4+3\sqrt{2}\right)\left(\sqrt{6}+\sqrt{2}\right)}{2} = \frac{4\sqrt{6}+4\sqrt{2}+6\sqrt{3}+6}{2} = 2\sqrt{6} + 2\sqrt{2} + 3\sqrt{3} + 3$

15 $T_1 = \frac{1}{2} CD \cdot BD \sin CDB$

$T_2 = \frac{1}{2} CD \cdot AD \sin CDA$

Since $C\hat{D}A = 180^\circ - C\hat{D}B$, their sines are the same: $\sin C\hat{D}A = \sin C\hat{D}B$.

Hence: $\frac{T_1}{T_2} = \frac{\cancel{\frac{1}{2}} \cancel{CD} \cdot BD \cancel{\sin CDB}}{\cancel{\frac{1}{2}} \cancel{CD} \cdot AD \cancel{\sin CDA}} = \frac{BD}{AD}$.

16 a $60^\circ + \theta < 180^\circ \Rightarrow \theta < 120^\circ$; hence, $0^\circ < \theta < 120^\circ$.

b $A = \frac{1}{2} 30 \cdot KJ \sin \theta$; we have to determine KJ. Using the law of sines:

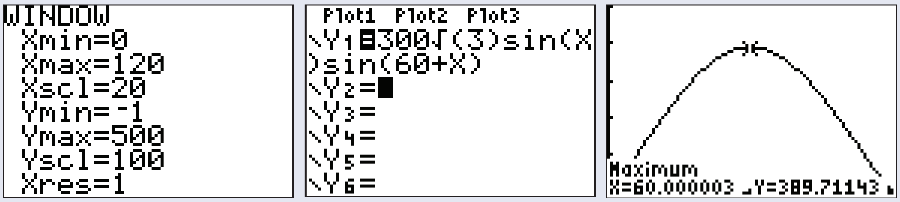
$$\frac{KJ}{\sin L} = \frac{30}{\sin 60^\circ} \Rightarrow KJ = \frac{30}{\frac{\sqrt{3}}{2}} \sin L$$

Since $L = 180 - (60^\circ + \theta) \Rightarrow \sin L = \sin(60^\circ + \theta)$, we have:

$KJ = \frac{30 \cdot 2}{\sqrt{3}} \sin\left(60^\circ + \theta\right) = 20\sqrt{3} \sin\left(60^\circ + \theta\right)$. Finally, the area is:

$$A = \frac{1}{2} 30 \cdot KJ \sin \theta = \frac{1}{2} 30 \cdot 20\sqrt{3} \sin\left(60^\circ + \theta\right) \sin \theta = 300\sqrt{3} \sin \theta \sin\left(60^\circ + \theta\right).$$

c



Hence, 60° gives the maximum area for the enclosure.

17 a $A = \frac{1}{2} 30 \cdot \sqrt{17^2 - 15^2} = 120 \text{ cm}^2$

b $\cos A\hat{B}C = \frac{2 \cdot 17^2 - 30^2}{2 \cdot 17^2} = -0.557\,09... \Rightarrow A\hat{B}C \approx 2.16$

c $R = A_{A_1} - \left(A_{A_2} - A_T\right)$

$$A_{A_1} = \frac{1}{2} 15^2 \pi \approx 353.43 \text{ cm}^2$$

$$A_{A2} = \frac{1}{2} r^2 \cdot A\hat{B}C \approx \frac{1}{2} 17^2 \cdot 2.16 = 312.12 \text{ cm}^2$$

$$R = A_{A_1} - A_{A_2} + A_T \approx 161 \text{ cm}^2$$

18 a Using the law of cosines, we have:

$$L^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos \alpha = 2 - 2\cos \alpha \Rightarrow L = \sqrt{2 - 2\cos \alpha}$$

b $\cos 2\left(\frac{\alpha}{2}\right) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right) \Rightarrow 2\sin^2\left(\frac{\alpha}{2}\right) = 1 - \cos \alpha \Rightarrow \sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1 - \cos \alpha}{2}}$

c $L = \sqrt{2 - 2\cos \alpha} = \sqrt{2(1 - \cos \alpha)} = \sqrt{4\left(\frac{1 - \cos \alpha}{2}\right)} = 2\sqrt{\frac{1 - \cos \alpha}{2}} = 2\sin\left(\frac{\alpha}{2}\right)$

19 Using the law of sines, we have:

$$\frac{a}{\sin \theta} = \frac{b}{\sin 2\theta}$$

Applying the double angle formula for sine, we have:

$$\frac{a}{\sin \theta} = \frac{b}{2\sin \theta \cos \theta} \Rightarrow a = \frac{b}{2\cos \theta} \Rightarrow \cos \theta = \frac{b}{2a}$$

Chapter 9

Practice questions

1 **a** $\overrightarrow{DC} = \overrightarrow{DM} + \overrightarrow{MC} = -\mathbf{u} + \mathbf{v} = \mathbf{v} - \mathbf{u}$

b $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}(\mathbf{v} - \mathbf{u})$

c $\overrightarrow{BC} = \overrightarrow{BM} + \overrightarrow{MC} = -\overrightarrow{AM} + \mathbf{v} = -\frac{1}{2}(\mathbf{v} - \mathbf{u}) + \mathbf{v} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$

d $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = (\mathbf{v} - \mathbf{u}) + \frac{1}{2}(\mathbf{u} + \mathbf{v}) = \frac{3}{2}\mathbf{v} - \frac{1}{2}\mathbf{u}$

2 **a** $\mathbf{w} = 2\mathbf{u} + \mathbf{v} = 2(\mathbf{i} - 2\mathbf{j}) + 4\mathbf{i} + 3\mathbf{j} = 6\mathbf{i} - \mathbf{j} = (6, -1)$

b $\mathbf{z} = \frac{6}{|\mathbf{w}|}\mathbf{w} = \frac{6}{\sqrt{6^2 + (-1)^2}}(6\mathbf{i} - \mathbf{j}) = \frac{6}{\sqrt{37}}(6\mathbf{i} - \mathbf{j}) = \frac{36}{\sqrt{37}}\mathbf{i} - \frac{6}{\sqrt{37}}\mathbf{j} = \frac{6}{\sqrt{37}}(6, -1)$

3 **a** $\left|\overrightarrow{OR}\right| = \sqrt{10^2 + (5\sqrt{5})^2} = 15$

Point R lies on the circle because the distance from O to R is 15 (the radius of the circle).

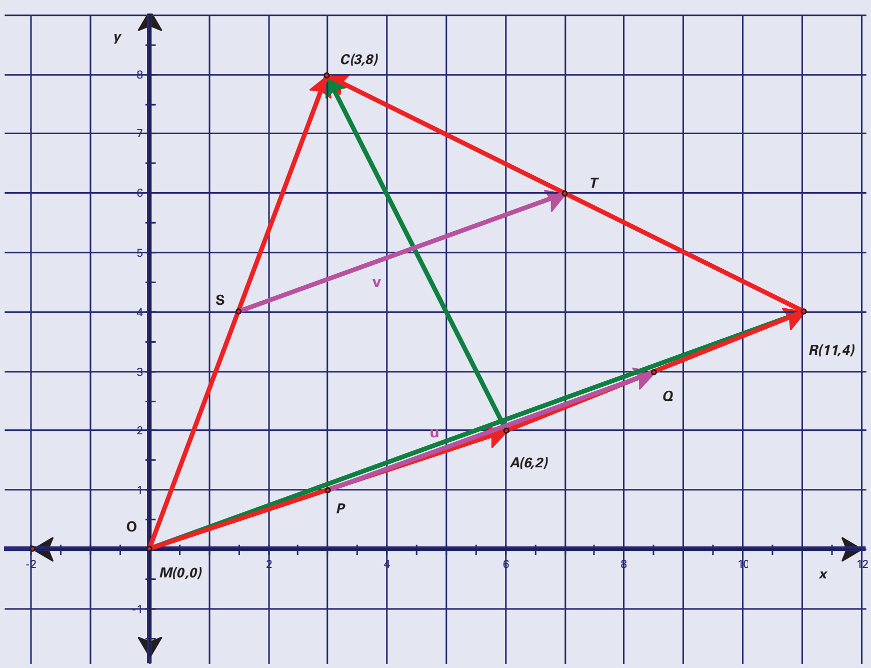
b $\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA} = \begin{pmatrix} 10 \\ 5\sqrt{5} \end{pmatrix} - \begin{pmatrix} 15 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 5\sqrt{5} \end{pmatrix}$

c $\cos \angle OAR = \frac{\overrightarrow{AO} \cdot \overrightarrow{AR}}{\left|\overrightarrow{AO}\right| \cdot \left|\overrightarrow{AR}\right|} = \frac{-15 \times (-5) + 0 \times 5\sqrt{5}}{15\sqrt{(-5)^2 + (5\sqrt{5})^2}} = \frac{75}{15\sqrt{150}} = \frac{1}{\sqrt{6}}$

d $\sin \angle OAR = \sqrt{1 - (\cos \angle OAR)^2} = \sqrt{1 - \left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{5}{6}}$

$\text{Area}_{\triangle MAR} = \frac{1}{2}\left|\overrightarrow{AM}\right| \cdot \left|\overrightarrow{AR}\right| \sin \angle OAR = \frac{1}{2} \cdot 30 \cdot \sqrt{150} \cdot \sqrt{\frac{5}{6}} = 75\sqrt{5}$

4



a $\overrightarrow{MR} = \begin{pmatrix} 11 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 \\ 4 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

b $\cos \measuredangle\left(\overrightarrow{MR}, \overrightarrow{AC}\right) = \dfrac{\overrightarrow{MR} \cdot \overrightarrow{AC}}{\left|\overrightarrow{MR}\right| \cdot \left|\overrightarrow{AC}\right|} = \dfrac{|11 \times (-3) + 4 \times 6|}{\sqrt{11^2 + 4^2} \cdot \sqrt{(-3)^2 + 6^2}} = \dfrac{9}{\sqrt{137} \cdot \sqrt{45}} \Rightarrow \measuredangle\left(\overrightarrow{MR}, \overrightarrow{AC}\right) = 83.4^\circ$

c The midpoints are $P(3, 1)$, $Q\left(\dfrac{17}{2}, 3\right)$, $S\left(\dfrac{3}{2}, 4\right)$, and $T(7, 6)$, so the vectors joining the midpoints are:

$$\mathbf{u} = \overrightarrow{PQ} = \begin{pmatrix} \frac{17}{2} \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ 2 \end{pmatrix}, \mathbf{v} = \overrightarrow{ST} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ 2 \end{pmatrix}$$

We can see that $\mathbf{u} = \mathbf{v} = \dfrac{1}{2}\overrightarrow{MR} \Rightarrow |\mathbf{u}| = |\mathbf{v}|$ and $\mathbf{u} \parallel \mathbf{v}$, so the points $PQST$ form the parallelogram.

5 $m(\mathbf{u} + \mathbf{v}) - 5\mathbf{i} + 7\mathbf{j} = n(\mathbf{u} - \mathbf{v})$

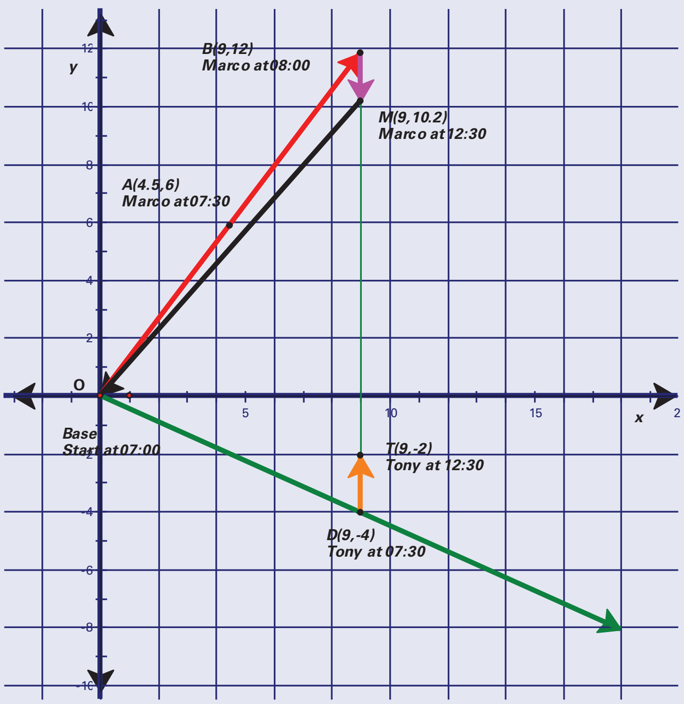
$m\left[(5\mathbf{i} + 3\mathbf{j}) + (\mathbf{i} - 4\mathbf{j})\right] - 5\mathbf{i} + 7\mathbf{j} = n\left[(5\mathbf{i} + 3\mathbf{j}) - (\mathbf{i} - 4\mathbf{j})\right]$

$m\left[6\mathbf{i} - \mathbf{j}\right] - 5\mathbf{i} + 7\mathbf{j} = n\left[4\mathbf{i} + 7\mathbf{j}\right]$

$(6m - 5)\mathbf{i} + (-m + 7)\mathbf{j} = 4n\mathbf{i} + 7n\mathbf{j}$

$$\begin{cases} 6m - 5 = 4n \\ -m + 7 = 7n \end{cases} \Rightarrow m = \frac{63}{46}, n = \frac{37}{46}$$

6



a The speed of 'Marco' is: $m = \sqrt{9^2 + 12^2} = 15$ km/h.

The speed of 'Tony' is: $t = \sqrt{18^2 + (-8)^2} \approx 19.7$ km/h.

b At 07:30 each crew will have been travelling for 0.5 hour, so their position vectors will be equal to one-half of their velocity vectors.

Position of 'Marco' is: $\overrightarrow{OA} = \dfrac{1}{2}\begin{pmatrix} 9 \\ 12 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 6 \end{pmatrix}$.

Position of 'Tony' is: $\overrightarrow{OD} = \dfrac{1}{2}\begin{pmatrix} 18 \\ -8 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \end{pmatrix}$.

c The distance between the vehicles at 07:30 is:

$$|\overrightarrow{AD}| = \left|\begin{pmatrix} 4.5 \\ -10 \end{pmatrix}\right| = \sqrt{4.5^2 + (-10)^2} \approx 10.97 \text{ km}.$$

d 'Marco' is directly north of 'Tony' when the x-coordinate of its position point is 9, which is one hour after they left the base port; i.e. 'Marco' starts work at 08:00 (and is at $B(9,12)$).

e The 'Marco' crew work for 4.5 hours and lay $4.5 \cdot 0.4 = 1.8$ km of pipe in a southerly direction.

The 'Tony' crew work for 5 hours and lay $5 \cdot 0.4 = 2$ km of pipe towards the north.

At 12:30 'Marco' is at point $M(9, 12 - 1.8) = M(9, 10.2)$ and 'Tony' is at point $T(9, -4 + 2) = T(9, -2)$.

The distance between them at this time is: $10.2 + 2 = 12.2$ km.

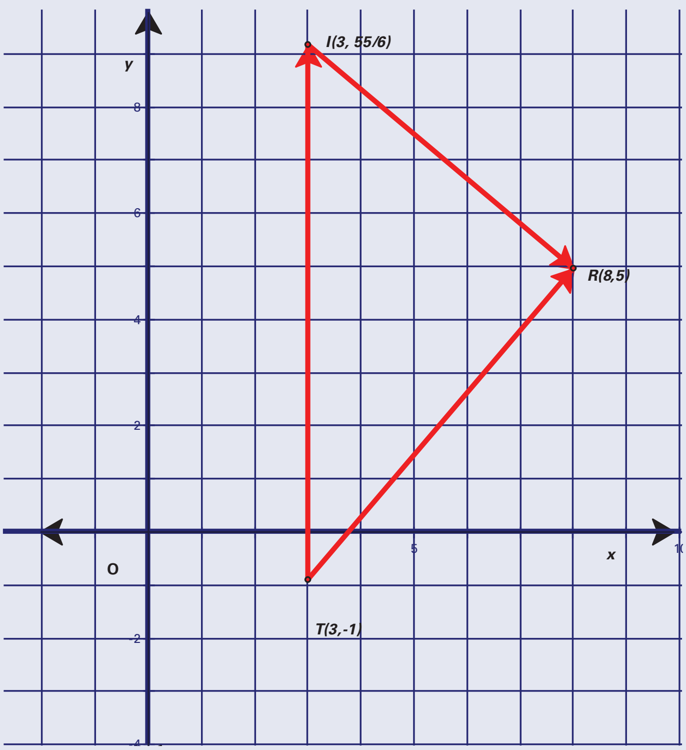
f The distance from M to base port for 'Marco' is: $|\overrightarrow{OM}| = \sqrt{9^2 + 10.2^2} \approx 13.6$ km.

As they travel at 15 km/h, it would take them $13.6 / 15 = 0.9069$ hours $= 54$ minutes to return to base.

7 a $\overrightarrow{OR} = \overrightarrow{OT} + \overrightarrow{TR} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

As $\overrightarrow{TI} = k\mathbf{j}$, points T and I have the same x-coordinate, i.e. $\overrightarrow{OI} = \begin{pmatrix} 3 \\ a \end{pmatrix}$.

$\overrightarrow{TR} \cdot \overrightarrow{IR} = 0 \Rightarrow \begin{pmatrix} 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 8-3 \\ 5-a \end{pmatrix} = 0 \Rightarrow 5 \times 5 + 6(5 - a) = 0 \Rightarrow a = \dfrac{55}{6}$



b $\overrightarrow{IR} = \overrightarrow{OR} - \overrightarrow{OI} = \begin{pmatrix} 8 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ \frac{55}{6} \end{pmatrix} = \begin{pmatrix} 5 \\ -\frac{25}{6} \end{pmatrix}$

8 **a** For $t = 2$, the position of the AUA plane is: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \end{pmatrix} + 2\begin{pmatrix} 360 \\ 480 \end{pmatrix} = \begin{pmatrix} 745 \\ 1000 \end{pmatrix}$.

b The speed of the plane is: $\left|\begin{pmatrix} 360 \\ 480 \end{pmatrix}\right| = \sqrt{360^2 + 480^2} = 600$ km/h.

c The position of the LH plane is given by: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -155 \\ 1300 \end{pmatrix} + t\begin{pmatrix} 480 \\ -360 \end{pmatrix}$.

The planes will collide if the following system of equations has a unique solution:

$$\begin{cases} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \end{pmatrix} + t\begin{pmatrix} 360 \\ 480 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -155 \\ 1300 \end{pmatrix} + t\begin{pmatrix} 480 \\ -360 \end{pmatrix} \end{cases} \Rightarrow$$

$$\begin{cases} 25 + 360t = -155 + 480t \Rightarrow t = 1.5 \\ 40 + 480t = 1300 - 360t \Rightarrow t = 1.5 \end{cases}$$

Therefore, the planes will collide after 1.5 hours, i.e. at 01:30.

d The position vector of the LH plane at $t = 1$ is: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -155 \\ 1300 \end{pmatrix} + 1\begin{pmatrix} 480 \\ -360 \end{pmatrix} = \begin{pmatrix} 325 \\ 940 \end{pmatrix}$.

e After 2 hours the LH plane is at: $\begin{pmatrix} 325 \\ 940 \end{pmatrix} + 1\begin{pmatrix} 450 \\ -390 \end{pmatrix} = \begin{pmatrix} 775 \\ 550 \end{pmatrix}$.

The distance between the planes after 2 hours is: $\left|\begin{pmatrix} 775 \\ 550 \end{pmatrix} - \begin{pmatrix} 745 \\ 1000 \end{pmatrix}\right| = \left|\begin{pmatrix} 30 \\ -450 \end{pmatrix}\right| = \sqrt{30^2 + (-450)^2} = 451$ km.

9 $\begin{pmatrix} 3n \\ 2n+3 \end{pmatrix} \perp \begin{pmatrix} 2n-1 \\ 4-2n \end{pmatrix} \Rightarrow \begin{pmatrix} 3n \\ 2n+3 \end{pmatrix} \cdot \begin{pmatrix} 2n-1 \\ 4-2n \end{pmatrix} = 0 \Rightarrow 3n(2n-1) + (2n+3)(4-2n) = 0 \Rightarrow 2n^2 - n + 12 = 0$

For this quadratic equation, the discriminant $\Delta = (-1)^2 - 4 \times 2 \times 12 = -95 < 0$, so there is no solution for n.

10 For vectors $\mathbf{a} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{b} = \sin\theta\mathbf{i} + \cos\theta\mathbf{j}$, we have:

$$\cos\measuredangle(\mathbf{a}, \mathbf{b}) = \frac{\cos\theta \cdot \sin\theta + \sin\theta\cos\theta}{\sqrt{\cos^2\theta + \sin^2\theta}\sqrt{\sin^2\theta + \cos^2\theta}} = \frac{2\sin\theta\cos\theta}{1 \cdot 1}$$

$$= \sin(2\theta) = \cos\left(\frac{\pi}{2} - 2\theta\right) \Rightarrow \alpha = \frac{\pi}{2} - 2\theta$$

11 For vectors $\mathbf{a} = \begin{pmatrix} x_a \\ y_a \end{pmatrix}, \mathbf{b} = \begin{pmatrix} x_b \\ y_b \end{pmatrix} \Rightarrow \mathbf{a} + \mathbf{b} = \begin{pmatrix} x_a + x_b \\ y_a + y_b \end{pmatrix}, \mathbf{a} - \mathbf{b} = \begin{pmatrix} x_a - x_b \\ y_a - y_b \end{pmatrix}$.

Then:

$$|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}| \Rightarrow$$

$$\sqrt{(x_a + x_b)^2 + (y_a + y_b)^2} = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$$

$$x_a^2 + 2x_a x_b + x_b^2 + y_a^2 + 2y_a y_b + y_b^2 = x_a^2 - 2x_a x_b + x_b^2 + y_a^2 - 2y_a y_b + y_b^2 \Rightarrow$$

$$2x_a x_b + 2y_a y_b = -2x_a x_b - 2y_a y_b \Rightarrow x_a x_b = -y_a y_b$$

$$\mathbf{a} \cdot \mathbf{b} = \frac{x_a x_b + y_a y_b}{\sqrt{x_a^2 + y_a^2}\sqrt{x_b^2 + y_b^2}} = \frac{x_a x_b - x_a x_b}{\sqrt{x_a^2 + y_a^2}\sqrt{x_b^2 + y_b^2}} = 0$$

Chapter 10

Practice questions

1 Method I:

From $(1-i)z = 1-3i$, we have: $z = \frac{1-3i}{1-i} = \frac{(1-3i)(1+i)}{1+1} = \frac{1+3+i(1-3)}{2} = 2-i$. Hence, $x = 2,\ y = -1$.

Method II:

We can solve the task using the notation $z = x + yi$. In this case, we have to solve a system of equations:

$(1-i)(x+yi) = 1-3i \Rightarrow$

$x + y + i(y-x) = 1-3i$

Hence: $\begin{cases} x + y = 1 \\ -x + y = -3 \end{cases} \Rightarrow \begin{array}{l} x = 2 \\ y = -1 \end{array}$

2 **a** Since w is the non-real solution of the equation $z^3 = 1$, then $w \neq 1$. Hence, we have:

$$1 + w + w^2 = \frac{(1+w+w^2)(1-w)}{1-w} = \frac{1-\overbrace{w^3}^{=1}}{1-w} = \frac{0}{1} = 0$$

Note: We can calculate the value for each possible w:

If $w = \text{cis}\left(\frac{2\pi}{3}\right)$, then $1 + w + w^2 = 1 + \text{cis}\left(\frac{2\pi}{3}\right) + \text{cis}\left(\frac{4\pi}{3}\right) = 1 + \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$

$$= 1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2} = 0$$

If $w = \text{cis}\left(\frac{4\pi}{3}\right)$, then $1 + w + w^2 = 1 + \text{cis}\left(\frac{4\pi}{3}\right) + \text{cis}\left(\frac{8\pi}{3}\right) = 1 + \text{cis}\left(\frac{4\pi}{3}\right) + \text{cis}\left(\frac{2\pi}{3}\right)$. This is the same as in the previous case; hence, the value is 0.

b $(wx + w^2y)(w^2x + wy) = \overbrace{w^3}^{=1}x^2 + \overbrace{w^4}^{=w}xy + w^2xy + \overbrace{w^3}^{=1}y^2$

$= x^2 + y^2 + (w + w^2)xy$

$= x^2 + y^2 + (-1)xy$ Since $w + w^2 = -1$ (using the result from **a**).

$= x^2 + y^2 - xy$

3 **a** $(1+i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$

b Let $P(n)$ be the statement: $(1+i)^{4n} = (-4)^n$.

Basic step:

The basis step must be $P(1)$.

$(1+i)^4 = \left((1+i)^2\right) = (2i)^2 = 2^2 i^2 = -4 = (-4)^1$; hence, $P(1)$ is true.

Inductive step:

Assume that for some $k \in \mathbb{N}^+$, $P(k)$ is true.

$P(k)$... $(1+i)^{4k} = (-4)^k$

Now, $(1+i)^{4(k+1)} = (1+i)^{4k+4} = (1+i)^k (1+i)^4 = (-4)^k(-4) = (-4)^{k+1}$.

Hence, $P(k+1)$ is true, and, by mathematical induction, $P(n)$ is true for all $k \in \mathbb{N}^+$.

c $(1+i)^{32} = (1+i)^{4\cdot 8} = (-4)^8 = 65\,536$

4 **a** For z_1: $|z_1| = \sqrt{\frac{6}{4}+\frac{2}{4}} = \sqrt{2}$, $\tan\theta = -\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{6}}{2}} = -\frac{1}{\sqrt{3}}$, the fourth quadrant, $\theta = -\frac{\pi}{6}$; hence,

$$z_1 = \sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

For z_2: $|z_2| = \sqrt{1+1} = \sqrt{2}$, $\tan\theta = -\frac{1}{1} = -1$, the fourth quadrant, $\theta = -\frac{\pi}{4}$; hence,

$$z_2 = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

b $$\frac{z_1}{z_2} = \frac{\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)}{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)} = 1\left(\cos\left(-\frac{\pi}{6}+\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{6}+\frac{\pi}{4}\right)\right) = \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)$$

c $$\frac{z_1}{z_2} = \frac{\frac{\sqrt{6}-i\sqrt{2}}{2}}{1-i}\cdot\frac{1+i}{1+i} = \frac{\frac{\sqrt{6}-\sqrt{2}i^2+\sqrt{6}i-i\sqrt{2}}{2}}{1+1} = \frac{\sqrt{6}+\sqrt{2}+i\left(\sqrt{6}-\sqrt{2}\right)}{4}$$

Hence, $a = \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$, $b = \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$.

5 $$\left(\frac{z_1}{z_3}\right)^3 = \left(\frac{a}{b}\left(\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\right)\right)^3 = \left(\frac{a}{b}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)\right)^3$$

$$= \frac{a^3}{b^3}\left(\cos\left(-\frac{\pi}{12}\cdot 3\right) + i\sin\left(-\frac{\pi}{12}\cdot 3\right)\right) = \frac{a^3}{b^3}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \frac{a^3}{b^3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

Hence, $\left(\frac{z_1}{z_3}\right)^3 = \frac{a^3\sqrt{2}}{2b^3} - i\frac{a^3\sqrt{2}}{2b^3}$.

Note: We can obtain the same result by firstly raising to the cube power and then dividing. However, be sure that you express the number in the form $x + yi$, which means that you multiply by $\frac{a^3}{b^3}$. You can write the result in the form: $x = \frac{a^3\sqrt{2}}{2b^3}$, $y = -\frac{a^3\sqrt{2}}{2b^3}$ (or any equivalent form).

6 Let $z = x + yi$. Then:

$$\sqrt{(x+16)^2 + y^2} = 4\sqrt{(x+1)^2 + y^2} \Rightarrow (x+16)^2 + y^2 = 16\left((x+1)^2 + y^2\right)$$

$$x^2 + 32x + 256 + y^2 = 16x^2 + 32x + 16 + 16y^2$$

$$15x^2 + 15y^2 = 240 \Rightarrow x^2 + y^2 = 16$$

Hence, $|z| = \sqrt{x^2 + y^2} = \sqrt{16} = 4$.

7 Method I:

$$a + bi = \frac{5-i}{2-i} = \frac{5-i}{2-i}\cdot\frac{2+i}{2+i} = \frac{10+1+i(5-2)}{4+1} = \frac{11+3i}{5}$$

Therefore, $a = \frac{11}{5}$, $b = \frac{3}{5}$.

Note: If this is a Paper 2 task, you can perform the division using a GDC.

Method II:

We can find the solution using multiplication: $(2-i)(a+bi)=5-i \Rightarrow 2a+b+i(2b-a)=5-i$

Therefore:

$$\begin{cases}2a+b=5\\-a+2b=-1\end{cases} \Rightarrow a=\frac{11}{5},\, b=\frac{3}{5}$$

8 Method I:

If $\arg(b+i)^2=60^\circ \Rightarrow \arg(b+i)=30^\circ \Rightarrow \tan 30^\circ=\frac{1}{b} \Rightarrow \frac{1}{\sqrt{3}}=\frac{1}{b} \Rightarrow b=\sqrt{3}$

Method II:

$$\arg(b+i)^2=60^\circ \Rightarrow \arg\left(b^2-1+2bi\right)=30^\circ \Rightarrow \tan 30^\circ=\frac{2b}{b^2-1} \Rightarrow \sqrt{3}=\frac{2b}{b^2-1}$$

$$\sqrt{3}b^2-2b-\sqrt{3}=0 \Rightarrow b_1=\sqrt{3},\, b_2=-\frac{1}{\sqrt{3}}$$

Since b is positive, the solution is $b=\sqrt{3}$.

9 $i(z+2)=1-2z \Rightarrow zi+2i=1-2z \Rightarrow zi+2z=1-2i \Rightarrow z(2+i)=1-2i$

$$z=\frac{1-2i}{2+i}=\frac{1-2i}{2+i}\cdot\frac{2-i}{2-i}=\frac{2-2+i(-1-4)}{4+1}=\frac{-5i}{5}=-i$$

10 a $z^5-1=(z-1)\left(z^4+z^3+z^2+z+1\right)$

b $z^5=1=\text{cis}\,0$

Hence, the zeros are the fifth roots of unity:

$$\text{cis}\left(\frac{2k\pi}{5}\right);\, k=0,1,2,3,4$$

$$1,\ \text{cis}\left(\frac{2\pi}{5}\right),\ \text{cis}\left(\frac{4\pi}{5}\right),\ \underbrace{\text{cis}\left(\frac{6\pi}{5}\right)}_{\text{cis}\left(-\frac{4\pi}{5}\right)},\ \underbrace{\text{cis}\left(\frac{8\pi}{5}\right)}_{\text{cis}\left(-\frac{2\pi}{5}\right)}.$$

The solutions are: $1,\ \text{cis}\left(\pm\frac{2\pi}{5}\right),\ \text{cis}\left(\pm\frac{4\pi}{5}\right)$.

c
$$\begin{aligned}\left(z-\text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z-\text{cis}\left(-\frac{2\pi}{5}\right)\right)&=\left(z-\cos\left(\frac{2\pi}{5}\right)-i\sin\left(\frac{2\pi}{5}\right)\right)\left(z-\cos\left(-\frac{2\pi}{5}\right)-i\sin\left(-\frac{2\pi}{5}\right)\right)\\&=\left(z-\cos\left(\frac{2\pi}{5}\right)-i\sin\left(\frac{2\pi}{5}\right)\right)\left(z-\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)\right)\\&=\left(z-\cos\left(\frac{2\pi}{5}\right)\right)^2+\sin^2\left(\frac{2\pi}{5}\right)\\&=z^2-2\cos\left(\frac{2\pi}{5}\right)z+\cos^2\left(\frac{2\pi}{5}\right)+\sin^2\left(\frac{2\pi}{5}\right)=z^2-2\cos\left(\frac{2\pi}{5}\right)z+1\end{aligned}$$

In the same way:

$$\begin{aligned}\left(z-\text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z-\text{cis}\left(-\frac{2\pi}{5}\right)\right)&=\left(z-\cos\left(\frac{4\pi}{5}\right)-i\sin\left(\frac{4\pi}{5}\right)\right)\left(z-\cos\left(-\frac{4\pi}{5}\right)-i\sin\left(-\frac{4\pi}{5}\right)\right)\\&=\left(z-\cos\left(\frac{2\pi}{5}\right)-i\sin\left(\frac{2\pi}{5}\right)\right)\left(z-\cos\left(\frac{4\pi}{5}\right)+i\sin\left(\frac{4\pi}{5}\right)\right)\\&=\left(z-\cos\left(\frac{4\pi}{5}\right)\right)^2+\sin^2\left(\frac{4\pi}{5}\right)=z^2-2\cos\left(\frac{4\pi}{5}\right)z+1\end{aligned}$$

Hence: $z^4+z^3+z^2+z+1=\left(z^2-2\cos\left(\frac{2\pi}{5}\right)z+1\right)\left(z^2-2\cos\left(\frac{4\pi}{5}\right)z+1\right)$

11 a $|8i| = 8$, $\tan\theta$ is not defined, positive y-axis, $\theta = \frac{\pi}{2}$, $8i = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

b i $$\sqrt[3]{8i} = \sqrt[3]{8}\left(\cos\left(\frac{\frac{\pi}{2}}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}}{3} + \frac{2k\pi}{3}\right)\right) = \sqrt[3]{8}\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right);\ k = 0, 1, 2$$

For $k = 0$, the number is in the first quadrant: $\sqrt[3]{8i} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\operatorname{cis}\frac{\pi}{6}$

ii $$\sqrt[3]{8i} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

12 a i All the numbers are of modulus 1; hence, their product and quotient is of modulus 1 and $|z| = 1$.

ii $$z = \frac{\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)^2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)^3}{\left(\cos\left(\frac{\pi}{24}\right) + i\sin\left(\frac{\pi}{24}\right)\right)^4}$$

$$= \frac{\left(\cos\left(-2\cdot\frac{\pi}{4}\right) + i\sin\left(-2\cdot\frac{\pi}{4}\right)\right)\left(\cos\left(3\cdot\frac{\pi}{3}\right) + i\sin\left(3\cdot\frac{\pi}{3}\right)\right)}{\left(\cos\left(4\cdot\frac{\pi}{24}\right) + i\sin\left(4\cdot\frac{\pi}{24}\right)\right)}$$

$$= \frac{\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)(\cos(\pi) + i\sin(\pi))}{\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)}$$

Hence, $\arg z = -\frac{\pi}{2} + \pi + \frac{\pi}{6} = \frac{2\pi}{3}$.

b Since $z = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$, then

$$z^3 = \left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^3 = \cos\left(3\cdot\frac{2\pi}{3}\right) + i\sin\left(3\cdot\frac{2\pi}{3}\right) = \underbrace{\cos(2\pi)}_{=1} + i\underbrace{\sin(2\pi)}_{=0} = 1$$

Hence, z is a cube root of one.

c $$(1+2z)(2+z^2) = 2 + 4z + z^2 + 2\underbrace{z^3}_{=1} = 4 + 4z + z^2 = 4 + 4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) + \left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^2$$

$$= 4 + 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = 4 - 2 + 2\sqrt{3}i - \frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

Note: We can write $4 + 4z + z^2$ as $1 + z + z^2 + 3 + 3z$. Using the property of a root of one, $1 + z + z^2 = 0$, we have:

$$4 + 4z + z^2 = 3 + 3z = 3 + 3\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = 3 + 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

13 $$\sqrt{z} = \frac{2}{1-i} + 1 - 4i = \frac{2}{1-i}\cdot\frac{1+i}{1+i} + 1 - 4i = \frac{2+2i}{1+1} + 1 - 4i = 1 + i + 1 - 4i = 2 - 3i$$

$$z = (2-3i)^2 = 4 - 12i - 9 = -5 - 12i$$

14 a Let $P(n)$ be the statement: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Basis step:

The basis step is $P(1)$ and it is true, because both sides are $\cos\theta + i\sin\theta$.

Inductive step:

Assume that for some $k \in \mathbb{N}^+$ $P(k)$ is true.

$P(k) \ldots (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Now, $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$$

$$= \cos(k\theta + \theta) + i(\sin(k\theta + \theta)) = \cos((k+1)\theta) + i\sin((k+1)\theta)$$

Therefore, $P(k+1)$ is true and, by mathematical induction, $P(n)$ is true for all $k \in \mathbb{N}^+$.

b i Method I:

$$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta}\cdot\frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos(-\theta) + i\sin(-\theta)$$

Method II:

Using de Moivre's theorem: $\frac{1}{z} = z^{-1} = (\cos(\theta) + i\sin(\theta))^{-1} = \cos(-\theta) + i\sin(-\theta)$

Note: In this part we **can't** use the result from **a**, since here $n = -1$ and in **a** $n \in \mathbb{N}^+$.

ii $z^{-n} = \left(z^{-1}\right)^n = (\cos(-\theta) + i\sin(-\theta))^n = \cos(-n\theta) + i\sin(-n\theta)$

$z^n = (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$

$z^n + z^{-n} = \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta) = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta) = 2\cos(n\theta)$

c i $\left(z + z^{-1}\right)^5 = z^5 + 5z^4z^{-1} + 10z^3z^{-2} + 10z^2z^{-3} + 5zz^{-4} + z^{-5} = z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$

ii Using the result from **i**, we have: $\left(z + z^{-1}\right)^5 = (2\cos\theta)^5 = 32\cos^5\theta$

and $\left(z^5 + z^{-5}\right) + 5\left(z^3 + z^{-3}\right) + 10\left(z + z^{-1}\right) = 2\cos(5\theta) + 5\cdot 2\cos(3\theta) + 10\cdot 2\cos(\theta)$

Thus, $32\cos^5\theta = 2(\cos(5\theta) + 5\cos(3\theta) + 10\cos(\theta))$.

Therefore, $\cos^5\theta = \frac{1}{16}(\cos(5\theta) + 5\cos(3\theta) + 10\cos(\theta))$, and $a = 1, b = 5, c = 10$.

15 $2p + 2iq = q - ip - 2(1 - i) \Rightarrow 2p + 2iq = -2 + q + i(2 - p)$

Hence: $\begin{cases} 2p = -2 + q \\ 2q = 2 - p \end{cases} \Rightarrow \begin{array}{c} 2p - q = -2 \\ p + 2q = 2 \end{array} \Rightarrow p = -0.4, q = 1.2$

16 i $z_1^5 = \left(2\left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)\right)^5 = 2^5\left(\cos\left(5\cdot\frac{2\pi}{5}\right) + i\sin\left(5\cdot\frac{2\pi}{5}\right)\right) = 32(\cos(2\pi) + i\sin(2\pi)) = 32$

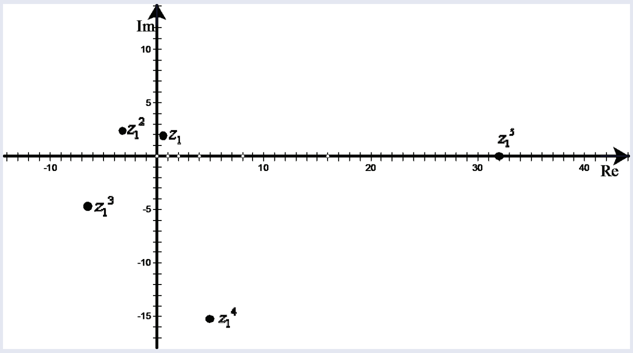
ii $z_1^2 = \left(2\left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)\right)^2 = 4\left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right) = 4\operatorname{cis}\frac{4\pi}{5}$

$z_1^3 = \left(2\left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)\right)^3 = 8\left(\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)\right) = 8\operatorname{cis}\frac{6\pi}{5}$

$z_1^4 = \left(2\left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)\right)^4 = 16\left(\cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)\right) = 16\operatorname{cis}\frac{\pi}{5}$

$z_1^5 = \left(2\left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)\right)^5 = 32(\cos(2\pi) + i\sin(2\pi)) = 32$

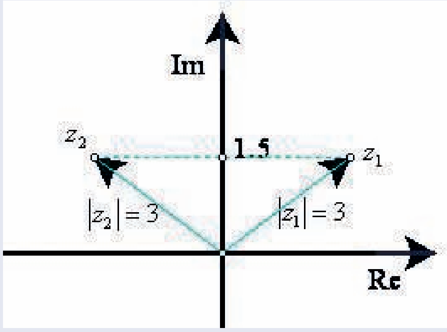
iii



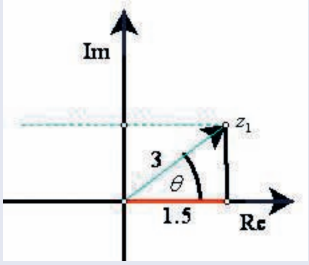
iv The transformation is a combination (in any order) of an enlargement of scale factor 2, with the origin as the centre, and an anti-clockwise rotation of $\frac{2\pi}{5}$, again with the origin as the centre.

17 a Let $z = a + bi$. Then: $\sqrt{a^2 + b^2} = \sqrt{a^2 + (b-3)^2} \Rightarrow a^2 + b^2 = a^2 + b^2 - 6b + 9 \Rightarrow 6b - 9 = 0 \Rightarrow b = \frac{3}{2}$

b i



ii Since $\arg z_1 = \theta$ and $\cos\theta = \frac{1.5}{3} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$.



iii $\arg z_2 = \pi - \arg z_1 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

c $\arg\left(\frac{z_1^k z_2}{2i}\right) = \arg\left(z_1^k\right) + \arg(z_2) - \arg(i) = \frac{k\pi}{6} + \frac{5\pi}{6} - \frac{\pi}{2} = \frac{k\pi}{6} + \frac{\pi}{3}$

Hence: $\frac{k\pi}{6} + \frac{\pi}{3} = \pi \Rightarrow k = 4$

18 $2a + b + i(2 - ab) = 7 - i \Rightarrow \begin{cases} 2a + b = 7 \\ 2 - ab = -1 \end{cases}$

Substituting $b = 7 - 2a$ (from the first equation) into the second equation:

$2 - a(7 - 2a) = -1 \Rightarrow 2a^2 - 7a + 3 = 0 \Rightarrow a_1 = \frac{1}{2}, a_2 = 3$

Since $a, b \in \mathbb{Z}$, the solution is $a = 3, b = 1$.

19 a $z^n = (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$

$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos(n\theta) - i\sin(n\theta)$

Hence, $z^n + \frac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta) = 2\cos(n\theta)$.

b $\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3\frac{1}{z} + 6z^2\frac{1}{z^2} + 4z\frac{1}{z^3} + \frac{1}{z^4} = z^4 + 4z^2 + 6 + 4\frac{1}{z^2} + \frac{1}{z^4} = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

Since $\left(z + \frac{1}{z}\right)^4 = (2\cos(\theta))^4$, $z^4 + \frac{1}{z^4} = 2\cos(4\theta)$, $z^2 + \frac{1}{z^2} = 2\cos(2\theta)$, we have:

$2^4\cos^4(\theta) = 2\cos(4\theta) + 4\cdot 2\cos(2\theta) + 6 \Rightarrow 8\cos^4(\theta) = \cos(4\theta) + 4\cos(2\theta) + 3$

Hence, $\cos^4(\theta) = \frac{1}{8}(\cos(4\theta) + 4\cos(2\theta) + 3)$.

20 a $z = \dfrac{\frac{1}{2}e^{2i\theta}}{e^{i\theta}} = \frac{1}{2}e^{i\theta}$

b $|z| = \frac{1}{2}$, so it is less than 1.

c Using the formula: $S_\infty = \dfrac{u_1}{1-r} = \dfrac{e^{i\theta}}{1-\left(\frac{1}{2}e^{i\theta}\right)}$

d i $S_\infty = \dfrac{\cos\theta + i\sin\theta}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}$

ii $S_\infty = e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots = \left(\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots\right) + i\left(\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 3\theta + \ldots\right)$

$\left(\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots\right) + i\left(\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 3\theta + \ldots\right) = \dfrac{\cos\theta + i\sin\theta}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}$

Taking the real parts, we have:

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \mathrm{Re}\left(\frac{\cos\theta + i\sin\theta}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}\right)$$

$$\mathrm{Re}\left(\frac{\cos\theta + i\sin\theta}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}\right) = \mathrm{Re}\left(\frac{\cos\theta + i\sin\theta}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}\cdot\frac{1 - \frac{1}{2}\cos\theta + i\frac{1}{2}\sin\theta}{1 - \frac{1}{2}\cos\theta + i\frac{1}{2}\sin\theta}\right)$$

$$= \frac{\cos\theta\left(1 - \frac{1}{2}\cos\theta\right) + i^2\frac{1}{2}\sin^2\theta}{\left(1 - \frac{1}{2}\cos\theta\right)^2 + \frac{1}{4}\sin^2\theta} = \frac{\cos\theta - \frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta}{1 - \cos\theta + \frac{1}{4}\cos^2\theta + \frac{1}{4}\sin^2\theta}$$

$$\frac{\cos\theta - \frac{1}{2}}{1 - \cos\theta + \frac{1}{4}} = \frac{4\cos\theta - 2}{5 - 4\cos\theta}$$

21 Method I:

If $-3 + 2i$ is a root, then $-3 - 2i$ is another root; therefore:

$P(z) = (z+2)(z+3-2i)(z+3+2i) = (z+2)\left((z+3)^2 - (2i)^2\right) = (z+2)\left(z^2 + 6z + 13\right) = z^3 + 8z^2 + 25z + 26$

So, $a = 8, b = 25, c = 26$.

Method II:

$P(-2) = 0 \Rightarrow -8 + 4a - 2b + c = 0$

$P(-3+2i) = 0 \Rightarrow 9 + 46i + a(5-12i) + b(-3+2i) + c = 0 \Rightarrow (9 + 5a - 3b + c) + (46 - 12a + 2b)i = 0$

Hence, we have to solve the system of equations:

$$\begin{cases} 4a - 2b + c = 8 \\ 5a - 3b + c = -9 \\ -12a + 2b = -46 \end{cases} \Rightarrow a = 8, b = 25, c = 26$$

22 Let $z = a + bi$:

$\sqrt{a^2 + b^2} = 2\sqrt{5} \Rightarrow a^2 + b^2 = 20$

$\dfrac{25}{a+bi} - \dfrac{15}{a-bi} = 1 - 8i \Rightarrow \dfrac{25(a-bi) - 15(a+bi)}{(a+bi)(a-bi)} = 1 - 8i$

$\dfrac{10a - 40bi}{a^2 + b^2} = 1 - 8i \Rightarrow \dfrac{10a - 40bi}{20} = 1 - 8i$

$10a - 40bi = 20 - 160i \Rightarrow a = 2, b = 4$

Hence, $z = 2 + 4i$.

23 $\begin{cases} iz_1 + 2z_2 = 3 \\ z_1 + (1-i)z_2 = 4 \;/\cdot(-i) \end{cases}$

$\begin{cases} iz_1 + 2z_2 = 3 \\ -iz_1 - (1+i)z_2 = -4i \end{cases}$

$(2 - 1 - i)z_2 = 3 - 4i \Rightarrow z_2 = \dfrac{3-4i}{1-i} = \dfrac{7}{2} - \dfrac{1}{2}i$

Substituting in the second equation: $z_1 + 3 - 4i = 4 \Rightarrow z_1 = 1 + 4i$

24 a $z_{1,2} = \dfrac{4 \pm \sqrt{16-32}}{2} = \dfrac{4 \pm 4i}{2} = 2 \pm 2i \Rightarrow z_1 = 2 + 2i, z_2 = 2 - 2i$

$|z_1| = \sqrt{4+4} = 2\sqrt{2}$, $\tan\theta = 1$, in the first quadrant, $\theta = \dfrac{\pi}{4}$; hence, $z_1 = 2\sqrt{2}e^{i\frac{\pi}{4}}$

$|z_2| = \sqrt{4+4} = 2\sqrt{2}$, $\tan\theta = -1$, in the fourth quadrant, $\theta = -\dfrac{\pi}{4}$; hence, $z_1 = 2\sqrt{2}e^{-i\frac{\pi}{4}}$

b $\dfrac{z_1^4}{z_2^2} = \dfrac{(2\sqrt{2})^4 e^{i\frac{4\pi}{4}}}{(2\sqrt{2})^2 e^{i\frac{-2\pi}{4}}} = 8e^{i\frac{3\pi}{2}} = 8\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right) = -8i$

c $z_1^4 = (2\sqrt{2})^4 e^{i\frac{4\pi}{4}} = 64e^{i\pi} = -64$

$z_2^4 = (2\sqrt{2})^4 e^{i\frac{-4\pi}{4}} = 64e^{-i\pi} = -64$

Thus, they are the same.

d $\dfrac{z_1}{z_2} + \dfrac{z_2}{z_1} = \dfrac{2\sqrt{2}e^{i\frac{\pi}{4}}}{2\sqrt{2}e^{i\frac{-\pi}{4}}} + \dfrac{2\sqrt{2}e^{i\frac{-\pi}{4}}}{2\sqrt{2}e^{i\frac{\pi}{4}}} = e^{i\left(\frac{\pi}{4}+\frac{\pi}{4}\right)} + e^{i\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)} = e^{i\frac{\pi}{2}} + e^{i\frac{-\pi}{2}} = i - i = 0$

e $z_1^n = (2\sqrt{2})^n e^{i\frac{n\pi}{4}} = 2^{\frac{3n}{2}} e^{i\frac{n\pi}{4}}$

The number is real if $\dfrac{n\pi}{4} = k\pi \Rightarrow n = 4k, k \in \mathbb{Z}$.

25 a $z^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 = \cos\frac{7\cdot 2\pi}{7} + i\sin\frac{7\cdot 2\pi}{7} = \underbrace{\cos 2\pi}_{=1} + i\underbrace{\sin 2\pi}_{=0} = 1$; hence $z^7 - 1 = 0$.

b $(z-1)\left(z^6+z^5+z^4+z^3+z^2+z+1\right) = z^7+z^6+z^5+z^4+z^3+z^2+z-z^6-z^5-z^4-z^3-z^2-z-1 = z^7-1$

Using the result from **a**, we have: $0 = z^7 - 1 = (z-1)\left(z^6+z^5+z^4+z^3+z^2+z+1\right)$. Since $z \neq 1$, then $z-1 \neq 0$ and hence $z^6+z^5+z^4+z^3+z^2+z+1=0$.

c Using the result from **b**, we have:

$$0 = z^6+z^5+z^4+z^3+z^2+z+1 = \operatorname{cis}\left(\frac{12\pi}{7}\right)+\operatorname{cis}\left(\frac{10\pi}{7}\right)+\operatorname{cis}\left(\frac{8\pi}{7}\right)+\operatorname{cis}\left(\frac{6\pi}{7}\right)+\operatorname{cis}\left(\frac{4\pi}{7}\right)+\operatorname{cis}\left(\frac{2\pi}{7}\right)+1$$

$$\operatorname{cis}\left(\frac{12\pi}{7}\right)+\operatorname{cis}\left(\frac{10\pi}{7}\right)+\operatorname{cis}\left(\frac{8\pi}{7}\right)+\operatorname{cis}\left(\frac{6\pi}{7}\right)+\operatorname{cis}\left(\frac{4\pi}{7}\right)+\operatorname{cis}\left(\frac{2\pi}{7}\right) = -1$$

Hence, $\operatorname{Re}\left(\operatorname{cis}\left(\frac{12\pi}{7}\right)+\operatorname{cis}\left(\frac{10\pi}{7}\right)+\operatorname{cis}\left(\frac{8\pi}{7}\right)+\operatorname{cis}\left(\frac{6\pi}{7}\right)+\operatorname{cis}\left(\frac{4\pi}{7}\right)+\operatorname{cis}\left(\frac{2\pi}{7}\right)\right) = -1$

$$\operatorname{Re}\left(\operatorname{cis}\left(\frac{12\pi}{7}\right)+\operatorname{cis}\left(\frac{10\pi}{7}\right)+\operatorname{cis}\left(\frac{8\pi}{7}\right)+\operatorname{cis}\left(\frac{6\pi}{7}\right)+\operatorname{cis}\left(\frac{4\pi}{7}\right)+\left(\frac{2\pi}{7}\right)\right)$$

$$= \cos\left(\frac{12\pi}{7}\right)+\cos\left(\frac{10\pi}{7}\right)+\cos\left(\frac{8\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right)$$

Since $\cos\left(\frac{12\pi}{7}\right) = \cos\left(\frac{2\pi}{7}\right)$, $\cos\left(\frac{10\pi}{7}\right) = \cos\left(\frac{4\pi}{7}\right)$, $\cos\left(\frac{8\pi}{7}\right) = \cos\left(\frac{6\pi}{7}\right)$, we have:

$$\cos\left(\frac{12\pi}{7}\right)+\cos\left(\frac{10\pi}{7}\right)+\cos\left(\frac{8\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right) = 2\left(\cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right)\right)$$

Finally: $2\left(\cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right)\right) = -1 \Rightarrow \cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$

Chapter 11

Practice questions

1 **a** $\sum_{i=1}^{30} y_i = 360 \Rightarrow \mu = \frac{360}{30} = 12$

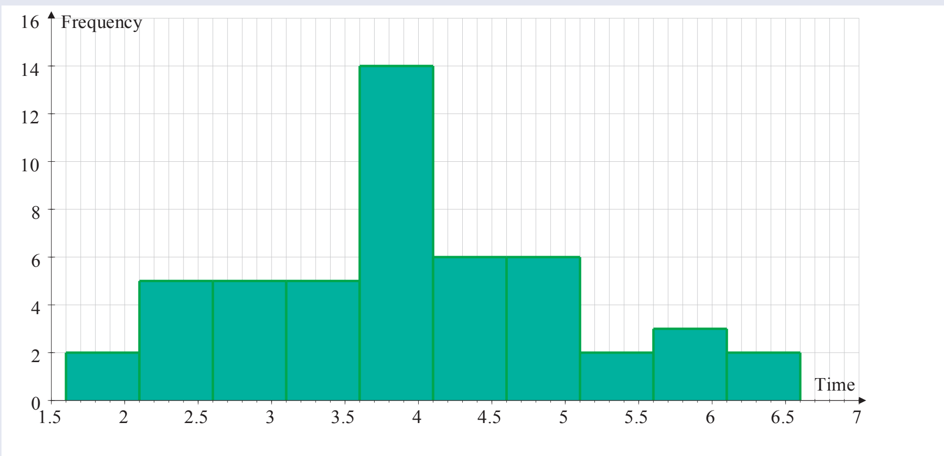
b $\sum_{i=1}^{30} (y_i - \mu)^2 = 925 \Rightarrow s_n = \sqrt{\frac{925}{30}} \approx 5.55$

2 $\mu = \frac{\sum x_i f_i}{\sum f_i} \Rightarrow 34 = \frac{10 \times 1 + 20 \times 2 + 30 \times 5 + 40 \times n + 50 \times 3}{11 + n} \Rightarrow 34 = \frac{350 + 40n}{11 + n} \Rightarrow$

$374 + 34n = 350 + 40n \Rightarrow 24 = 6n \Rightarrow n = 4$

3 **a**

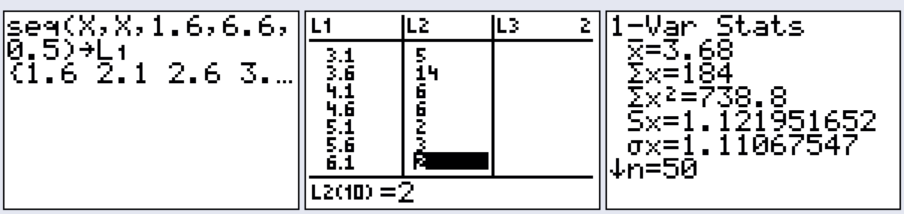
Time (minutes)	1.6	2.1	2.6	3.1	3.6	4.1	4.6	5.1	5.6	6.1	6.6
Frequency	2	5	5	5	14	6	6	2	3	2	0



b There are 7 out of 50 measurements that are greater than or equal to 5.1; therefore, the fraction of the measurements less than 5.1 is: $\frac{43}{50} = 0.86 = 86\%$.

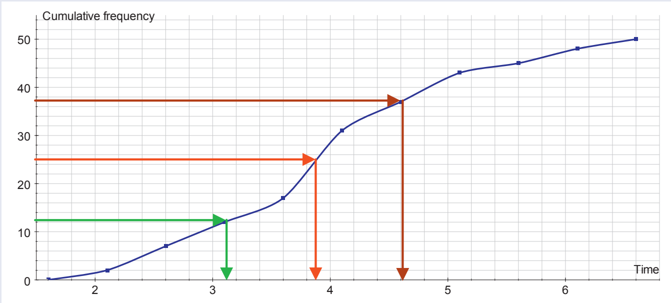
c There are 50 pieces of data, so, to determine the median, we need to find the 25th and 26th observations. We notice that these two observations are within the interval 3.6–4.1; therefore, the median is approximately 3.9.

d We can input the time as a sequence by using the sequence list feature on a GDC.



The mean value is 3.68, whilst the standard deviation is 1.11, correct to three significant figures.

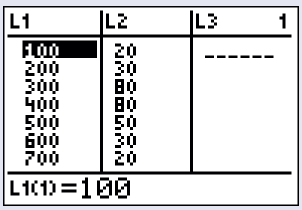
e

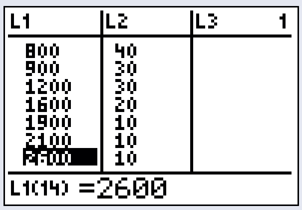


f Estimates for the minimum and maximum values are 1.6 and 6.6 respectively. The first and third quartiles correspond to the cumulative frequencies of 12.5 and 37.5 respectively; therefore, an estimate for the first quartile is 3.15 and the third quartile is 4.65. An estimate for the median (which corresponds to a cumulative frequency of 25) is 3.9.

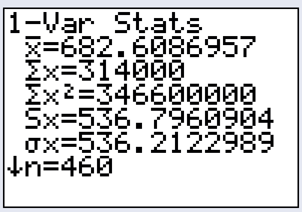
4 **a** The median and the IQR would best represent the data, since the data is skewed to the right and there are a few outliers on the right.

b Firstly, we are going to read the frequencies from the histogram and input them into the frequency distribution table on our GDC.



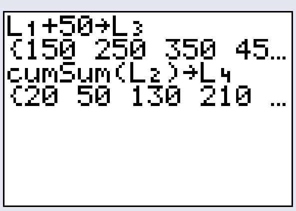


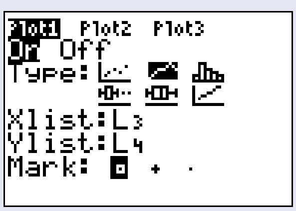
Finally, we apply the statistical calculation for one variable.

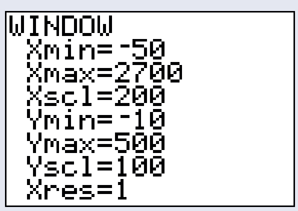


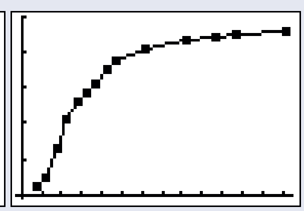
The mean value is 682.6 and the standard deviation is 536.2.

c Since we have grouped data, the endpoints of our intervals will be 150, 250, 350, ..., and so on. On a calculator, we can use the adding a number to the list feature. Alternatively, we can calculate the cumulative frequencies from the List menu.

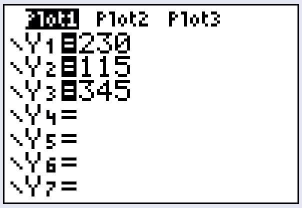


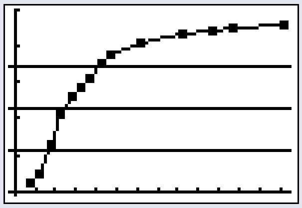




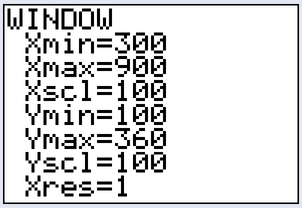


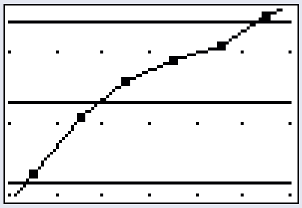
d There are 460 cities, so, to estimate the median, we need to draw a horizontal line from 230. To find the lower and upper quartiles, we need to draw horizontal lines from 115 and 345 respectively.





For a better estimation, we will use the zoom box feature and turn the grids on.





So, the median is about 500. The first quartile is about 330 and the third quartile is about 830. Therefore, the IQR is about 830 – 330 = 500.

e There are a few outliers to the right. The outliers are those points which are over $Q_3 + 1.5\text{IQR}$, i.e. $830 + 1.5 \cdot 500 = 1580$, which gives us 50 cities from the histogram.

f The data is skewed to the right with quite a few outliers to the right (1600 and above). The data is also bimodal, with the modal values being 300 and 400.

5 **a** It appears that Spain produces both the most expensive (estimated €152 per case) and the cheapest (estimated €55 per case) red wine.

b Red wines are generally more expensive in France as we can see that the median price is the highest; the minimum value in France is also the highest, but the upper 50% of wines are also within a very small range of approximately €10 per case.

c It appears that the wines are, on average, more expensive in France, where the prices are skewed towards the higher end. In Spain, you can find a higher percentage of cheaper wine than in the other two countries, but you also find the most expensive wines on the market; so Spain has the widest range of prices. Italy seems to have the most symmetrical distribution of wine prices.

6 **a** The mean value of the data is 52.6 and the standard deviation is 7.60, both given correctly to three significant figures.

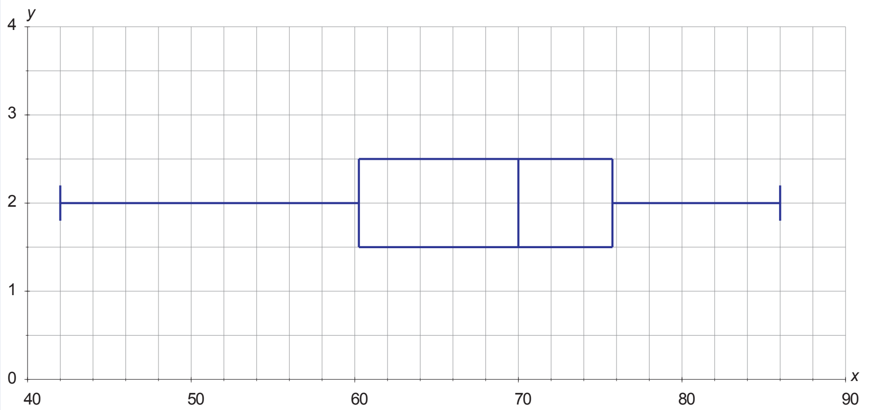
b The median value is 51.3. The upper and lower quartiles are 49.9 and 52.6 respectively; therefore, the IQR is 2.65, all correct to three significant figures.

c Since there is one outlier (112.72), the mean value is more influenced by it than the median value and IQR.

7 **a** The distribution is not symmetric since the median is not the midpoint between the minimum and maximum value, nor is it the midpoint between the first and the third quartile.

b The outliers lie 1.5IQR further to the left of the lower quartile and 1.5IQR further to the right of the upper quartile. If there are any outliers, they will lie below 37 and above 99, which is outside of the given range; therefore, there are no outliers.

c



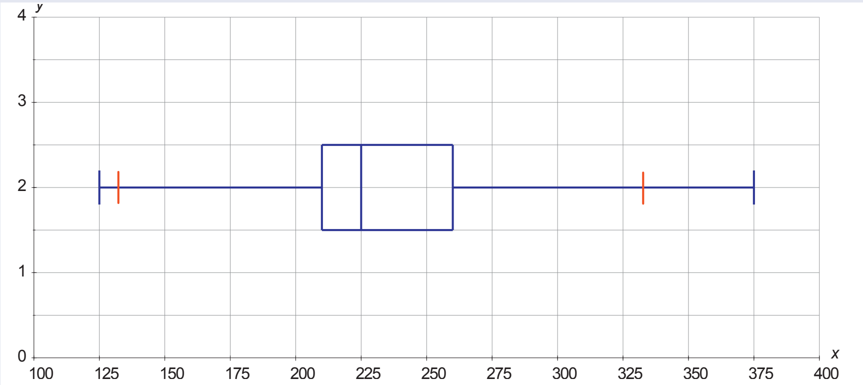
d Additionally to the description provided in part **a**, we can say that the data is skewed to the left.

8 **a** From the graph, we can estimate that the median cholesterol level (50% of the cumulative percentage) is 225.

b The first and third quartiles are estimated from 25% and 75% of the cumulative percentage; our estimations are 210 and 260 respectively. The 90th and 10th percentiles are estimated from 90% and 10% of the cumulative percentage; our estimations are 300 and 190 respectively.

c Using our answers from part **b**, the estimated IQR will be 50. Since 2000 patients have been studied, the number of patients in the middle 50%, ranging from 210 to 260, is 1000.

d

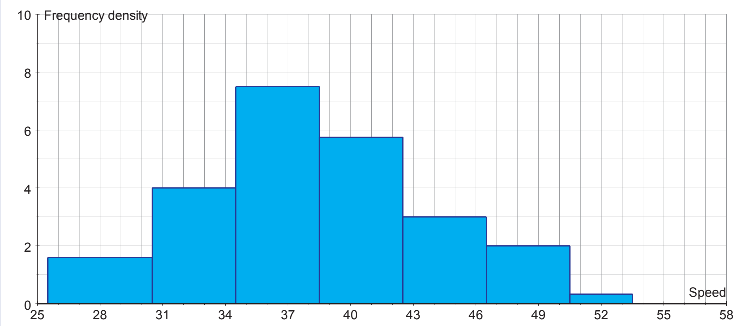


e We notice that the data is skewed to the right a bit, with more outliers on the right side, since the outliers lie outside of the interval 135 to 335. From the cumulative frequency graph, we read that there are almost 100 patients who have a cholesterol level greater than 335 mg/dl and only a few patients with a level less than 135 mg/dl.

9a

Speed	Frequency
26–30	10
31–34	16
35–38	30
39–42	23
43–46	12
47–50	8
51–54	1

b



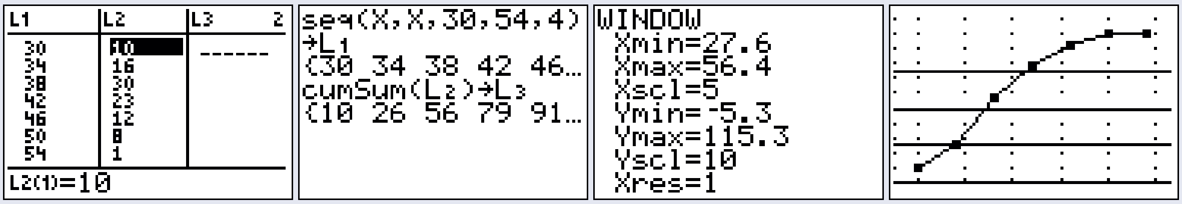
c

x_i	f_i	$x_i \times f_i$	$x_i^2 \times f_i$
28	10	280	7840
32.5	16	520	16 900
36.5	30	1095	39 967.5
40.5	23	931.5	37 725.75
44.5	12	534	23 763
48.5	8	388	18 818
52.5	1	52.5	2756.25
Σ	100	3801	147 770.5
	$\mu = 38.01$		$s^2 = 32.9449$
		$s = 5.739\,765$	

d

x_i	f_i	c_i
30	10	10
34	16	26
38	30	56
42	23	79
46	12	91
50	8	99
54	1	100

e In order to estimate the median, we need to draw the cumulative frequency diagram. We will do this using a calculator.



An estimate of the median is 37. Q_1 is 34 and Q_3 is 42; therefore, the IQR is 8.

f Since Q_3 is 42 and 1.5IQR is 12, there is a possible outlier, which is the largest observation of 54.

10 a

Classes	Frequency
1101–1200	1
1201–1300	1
1301–1400	0
1401–1500	1
1501–1600	3
1601–1700	5
1701–1800	11
1801–1900	3
1901–2000	11
2001–2100	8
2101–2200	4
2201–2300	2

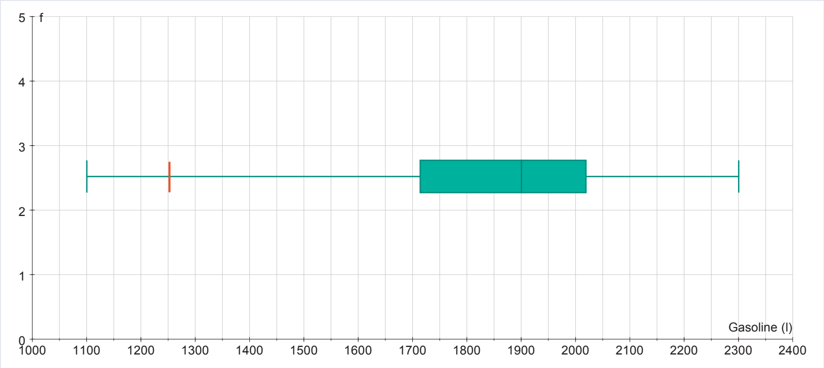
Using Autograph software with the grouped data, we obtain the following results:

The mean value is 1850.5 and the standard deviation is 232.379.

The median value is 1900.5. The first and third quartiles are 1714.14 and 2019.25 respectively; therefore, the IQR is 305.11.

b By looking at the frequency distribution table, we can surmise that there may be some outliers to the left, so we need to calculate $Q_1 - 1.5\text{IQR} = 1256$. From the table, we can see that we have one outlier.

c



d $\mu \pm s_n = 1850.5 \pm 232.379 \Rightarrow [1618, 2083]$

e Germany will definitely be an outlier since Q3 + 1.5IQR = 2477 < 2758; therefore, it will influence the mean and standard deviation. Conversely, the median, the first and third quartiles, and the IQR will not change much.

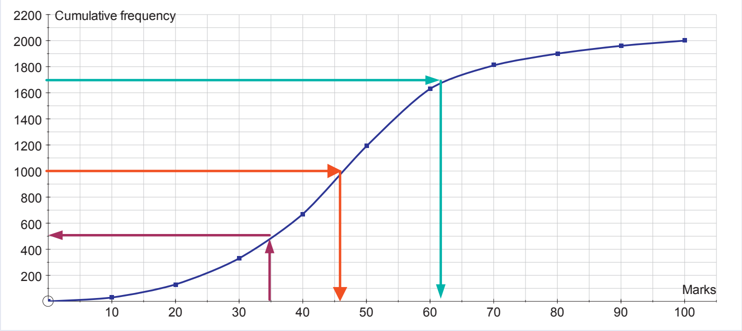
11 a $\sum_{i=1}^{90} x_i = 4460 \Rightarrow \mu = \dfrac{4460}{90} \approx 49.6$ minutes

b $\mu_{New} = \dfrac{4460 + 35 + 39 + 28 + 32}{90 + 4} = \dfrac{4594}{94} \approx 48.9$ minutes

12 a

Marks	$\leqslant 10$	$\leqslant 20$	$\leqslant 30$	$\leqslant 40$	$\leqslant 50$	$\leqslant 60$	$\leqslant 70$	$\leqslant 80$	$\leqslant 90$	$\leqslant 100$
Number of candidates	30	130	330	670	1190	1630	1810	1900	1960	2000

b



c i By looking at the graph, we estimate that the median score (which corresponds to the cumulative frequency of 1000) is 46.

ii We draw a vertical line from 35 on the Marks axis and reach the cumulative frequency curve at the point at which the cumulative frequency is about 500. Therefore, 500 candidates had to retake the exam.

iii The highest scoring 15% corresponds to the highest 300 results; therefore, we draw a horizontal line from 1700 on the Cumulative frequency axis and reach the curve at the point at which the number of marks is about 62. Hence, a distinction will be awarded if 62 or more marks are scored on the test.

13 $\mu = \dfrac{72 \times 1.79 + 28 \times 1.62}{72 + 28} = \dfrac{174.24}{100} \approx 1.74$ cm, correct to the nearest centimetre.

14 a $\sum_{i=1}^{25} x_i = 300 \Rightarrow \mu = \dfrac{300}{25} = 12$

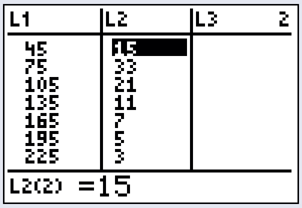
b $\sum_{i=1}^{25} (x_i - m)^2 = 625 \Rightarrow s_n = \sqrt{\dfrac{625}{25}} = 5$

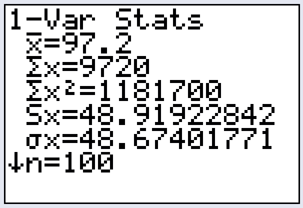
15

Score	10	20	30	40	50
Number of competitors	1	2	5	k	3

$$\bar{x} = 34 \Rightarrow \frac{10\times 1 + 20\times 2 + 30\times 5 + 40\times k + 50\times 3}{11+k} = 34 \Rightarrow 350 + 40k = 374 + 34k \Rightarrow 6k = 24 \Rightarrow k = 4$$

16 a To calculate an estimate for the mean, we will take the midpoints of the intervals (15, 45, 75, and so on) and the corresponding frequencies.





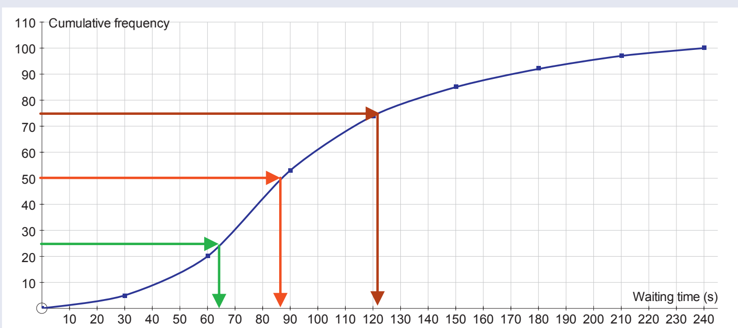
So, an estimate for the mean of the waiting times is 97.2 seconds.

(**Note:** Even though we could have used only the mean feature from the List menu, we used the whole statistics calculation since there is a chance that we will need further statistics in the following parts of the problem.)

b

Waiting time (seconds)	≤ 30	≤ 60	≤ 90	≤ 120	≤ 150	≤ 180	≤ 210	≤ 240
Cumulative frequency	5	20	53	74	85	92	97	100

c



d To find the three estimations asked for, we need to draw a horizontal line at 50 for the median and at 25 and 75 for the quartiles. An estimation of the mean value is 87, whilst the lower and upper quartiles are 65 and 121 respectively.

17 a Taking our readings from the histogram:

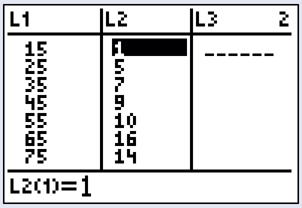
i There are 10 plants that have a length between 50 and 60 cm.

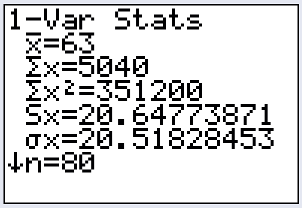
ii To find the number of trees that have a length between 70 and 90 cm, we need to add two frequencies: 14 + 10 = 24.

b As in question 15, we again take the midpoints of the intervals (15, 25, 35, ..., 95) and the corresponding frequencies.

x_i	15	25	35	45	55	65	75	85	95
f_i	1	5	7	9	10	16	14	10	8

We input the two lists into a GDC and get the following results.



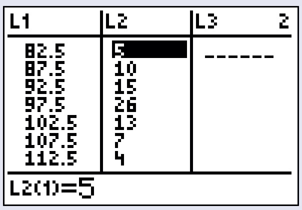


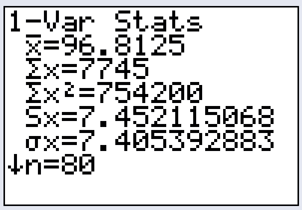
So, an estimate for the mean is 63, and the standard deviation is 20.5.

c The data is skewed to the left; therefore, the median and the mean are different values.

d Since there are 80 plants, the median corresponds to the cumulative frequency of 40. Since the values of the cumulative frequencies are symmetrical around 40, 32 for 60 and 48 for 70, we can estimate the cumulative frequency of 40 as 65, i.e. the median is less than or equal to 65 cm.

18 a Again, we will use the midpoints of the intervals (82.5, 87.5, 92.5, ..., and so on) and the corresponding frequencies. We put the two lists into our GDC and obtain an estimation of the standard deviation.



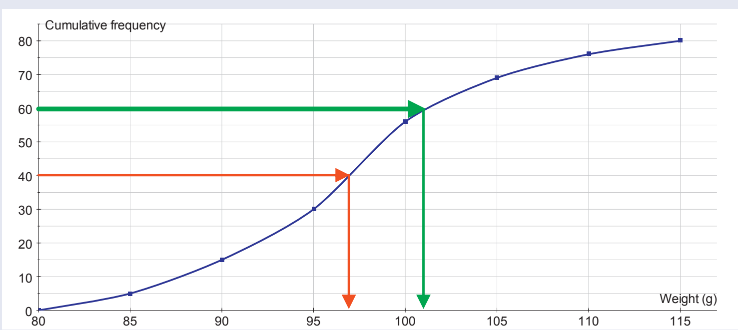


So, an estimate for the standard deviation of the weights is 7.405.

b

Weight (W)	$W \leqslant 85$	$W \leqslant 90$	$W \leqslant 95$	$W \leqslant 100$	$W \leqslant 105$	$W \leqslant 110$	$W \leqslant 115$
Number of packets	5	15	30	56	69	76	80

c



i An estimate for the median (which corresponds to the cumulative frequency of 40) is 97.

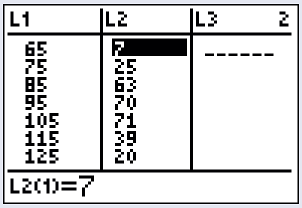
ii An estimate for the upper quartile (which corresponds to the cumulative frequency of 60) is 101.

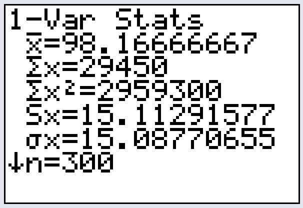
d $(W_1 - \overline{W}) + (W_2 - \overline{W}) + (W_3 - \overline{W}) + ... + (W_{80} - \overline{W}) = (W_1 + W_2 + W_3 + ... + W_{80}) - 80 \times \overline{W}$

$$= (W_1 + W_2 + W_3 + ... + W_{80}) - \cancel{80} \times \frac{W_1 + W_2 + W_3 + ... + W_{80}}{\cancel{80}} = 0$$

e There are 71 packets that satisfy the condition $85 < W \leqslant 110$. There are 20 packets that satisfy the condition $100 < W \leqslant 110$. Therefore, the probability is: $P(E) = \frac{20}{71} \approx 0.282$, correct to three significant figures.

19 a We will use the midpoints of the intervals (65, 75, 85, ..., and so on) and the corresponding frequencies. We put the two lists into our GDC and calculate the mean.



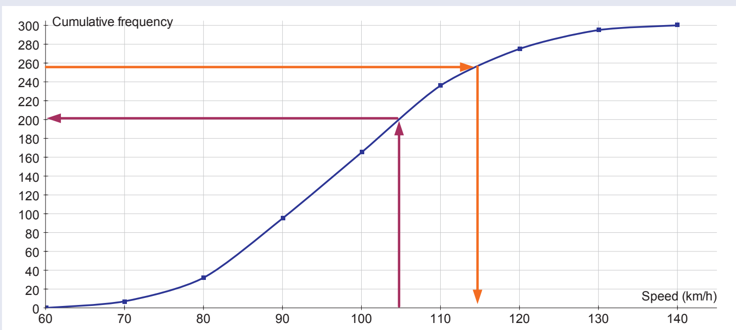


So, the mean speed is 98.2 km h^{-1}, correct to three significant figures.

b **i** To find the value of a we can either add 70 (the frequency of the speed interval 90–100) to the previous cumulative frequency, 95; or subtract 71 (the frequency of the speed interval 100–110) from the next cumulative frequency, 236. In both cases we get the same value: $a = 165$.

In a similar way, we find the value of b: $b = 236 + 39 = 275$.

ii



c **i** We draw a vertical line from $v = 105$ until we reach the cumulative frequency curve. This gives us an estimate for the cumulative frequency of 200. So, there are 100 cars that will exceed the speed of 105 km h^{-1} and

$P = \frac{100}{300} = 0.333... \approx 33.3\%$.

ii If 15% of the cars exceed this speed, then 85% do not exceed that speed. 85% of 300 is 255, so we draw a horizontal line from $y = 255$ until we reach the cumulative frequency curve. This gives us an estimate of a speed of 115 km h^{-1}.

20 a **i** We take a horizontal line across from 100 on the vertical axis until it touches the graph. From that point, we take a vertical line down to the horizontal axis and read off the value. An estimate for the median fare is \$24.

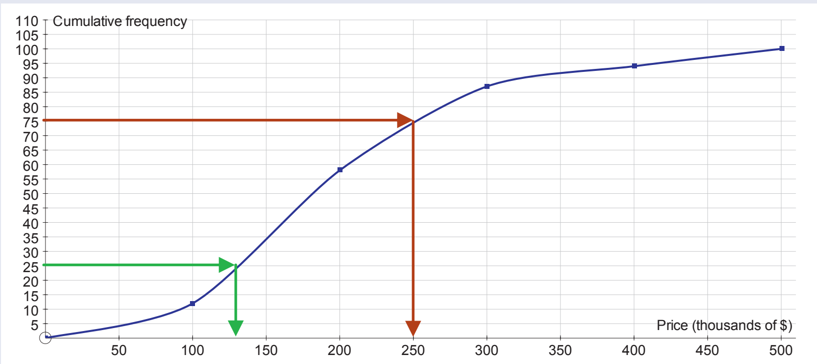
ii We take a vertical line up from 35 on the horizontal axis until we reach the graph. From that point, we take a horizontal line across to the vertical axis and read off the value. An estimate for the number of cabs in which the fare taken is \$35 or less is 158.

b 40% of the cabs is $0.4 \cdot 200 = 80$. So, we take a horizontal line from 80 on the vertical axis until we reach the graph and then we estimate the x-coordinate, which is 22. Therefore, the fare is \$22. To find the number of kilometres, we need to divide the fare by 0.55 (which is the fare per kilometre for distance travelled). Therefore, the distance travelled is 40 km.

c If the distance travelled is 90 km, the driver will earn $90 \cdot 0.55 = 49.5$ dollars. We will use the graph to estimate the number of cabs that will earn less than 49.5 dollars: there are 184, and therefore there are 16 cabs that will earn more than that. So, the percentage of the cabs that travel more than 90 km is: $\frac{16}{200} = 0.08 = 8\%$.

21 Since the three numbers are given in order of magnitude, $a < b < c$, we know that the middle one (the median) is 11, so $b = 11$. Given that the range is 10, we know that the difference between the minimum and maximum value is: $c - a = 10$. Since the mean value is 9, we can establish another equation in terms of a and c: $\frac{a+11+c}{3} = 9 \Rightarrow a + c = 16$. Solving these two equations (using the elimination method), we get: $2a = 6 \Rightarrow a = 3$. So, finally, $c = 13$.

22 a

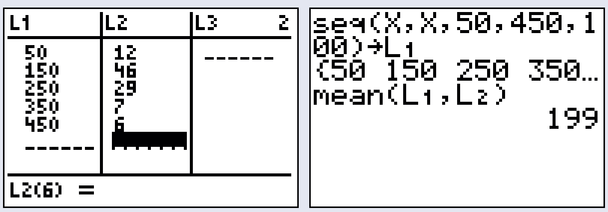


b By using horizontal lines at 25 and 75, we estimate the values of Q_1 and Q_3 as \$130 000 and \$250 000 respectively. Hence, the IQR is \$120 000.

c To find the frequencies a and b we need to subtract two successive cumulative frequencies.

$f_i = c_i - c_{i-1}$: $a = 94 - 87 = 7, b = 100 - 94 = 6$

d We input the midpoints of the intervals in the first list and the corresponding frequencies in the second list:



So, an estimate of the mean selling price is \$199 000.

e i An estimate of the cumulative frequency for \$350 000 is 92; therefore, there are about eight houses that can be described as *De Luxe*.

ii Out of eight *De Luxe* houses, six were sold for \$400 000; therefore, the probability that both selected houses have a selling price more than \$400 000 is:

$$P(E) = \frac{6}{8} \cdot \frac{5}{7} = \frac{15}{28}.$$

23 a i To mark the median, we draw the horizontal line $y = 40$ until it hits the graph, at which point we draw a vertical line down to the Diameter axis. An estimate for the median is 20 mm.

ii To mark the upper quartile, we draw the horizontal line $y = 60$ until it hits the graph, at which point we draw a vertical line down to the Diameter axis. An estimate for the upper quartile is 24 mm.

b The interquartile range is: IQR $= 24 - 14 = 10$ mm.

24 a In this question we can accept an error of ± 2 students.

We need to read the cumulative frequencies at the endpoints of the intervals and then subtract the successive ones to obtain the frequencies.

For 40 the cumulative frequency is 74, and therefore the corresponding frequency is $74 - 22 = 52$.

For 60 the cumulative frequency is 142, and therefore the corresponding frequency is $142 - 74 = 68$.

For 80 the cumulative frequency is 180, and therefore the corresponding frequency is $180 - 142 = 38$.

Mark (x)	$0 \leqslant x < 20$	$20 \leqslant x < 40$	$40 \leqslant x < 60$	$60 \leqslant x < 80$	$80 \leqslant x < 100$
Number of students	22	52	68	38	20

b 40% of 200 students is 80. The cumulative frequency of 80 corresponds to about 42 marks. So, the pass mark is about 42%.

25 a To find the median height we draw the horizontal line $y = 60$ until it hits the graph. We then estimate the x-coordinate of the point of intersection. We estimate 183 cm.

b For the lower and upper quartiles, we draw two horizontal lines: $y = 30$ and $y = 90$. Then we estimate the x-coordinates of the points of intersection. Therefore: $Q_1 = 175, Q_3 = 189 \Rightarrow \text{IQR} = 189 - 175 = 14$.

26 Since the modal value is 11, we know that $c = d = 11$. Given that the range is 8, we can find the value of a: $11 - a = 8 \Rightarrow a = 3$. Finally, given that the mean value is 8, we can find the remaining number b:

$$\frac{3 + b + 11 + 11}{4} = 8 \Rightarrow 25 + b = 32 \Rightarrow b = 7.$$

27 a We draw the vertical line $x = 40$ until it hits the graph. We then estimate the y-coordinate of the point of intersection as 100. So, the number of students who scored 40 marks or less is 100.

b There are 800 students, so the middle 50% is between 200 and 600 students. For a cumulative frequency of 200, the estimated mark is 55; whilst for 600, the estimated mark is 75. Hence, we say that the middle 50% of test results lie between 55 and 75 marks: $a = 55, b = 75$.

Solution Paper 1 type

28 We are going to add all the known observations and use the mean formula to find the first equation relating the unknowns: $13 = \dfrac{x + y + 90}{8} \Rightarrow 104 = x + y + 90 \Rightarrow x + y = 14$

We are going to use the sum of the squares of the known observations and the variance formula to find the second equation relating the squares of the unknowns:

$$21 = \frac{x^2 + y^2 + 1404}{8} - 13^2 \Rightarrow 190 = \frac{x^2 + y^2 + 1404}{8} \Rightarrow 1520 = x^2 + y^2 + 1404 \Rightarrow x^2 + y^2 = 116$$

Now, we need to solve the simultaneous equations by using the substitution method.

$$\begin{cases} x + y = 14 \\ x^2 + y^2 = 116 \end{cases} \Rightarrow \begin{cases} y = 14 - x \\ x^2 + (14 - x)^2 = 116 \end{cases} \Rightarrow \begin{cases} y = 14 - x \\ 2x^2 - 28x - 80 = 0 \end{cases} \Rightarrow \begin{cases} y = 14 - x \\ x^2 - 14x - 40 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} y = 14 - x \\ (x - 4)(x - 10) = 0 \end{cases} \Rightarrow \begin{cases} y = 14 - x \\ x = 4 \text{ or } x = 10 \end{cases} \Rightarrow \begin{cases} y = 10 \text{ or } \cancel{y = 4} \\ x = 4 \text{ or } \cancel{x = 10} \end{cases}$$

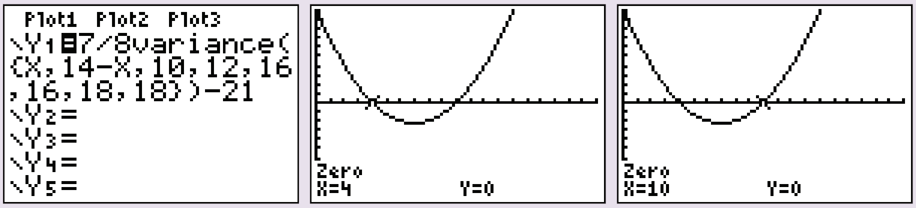
Since $x < y$, we can discard the second solution.

Solution Paper 2 type

28 Since the calculator cannot solve simultaneous equations by using the list features, we need to reduce the equation to only one unknown. Therefore, the mean value is going to give us the substitution.

$$13 = \frac{x + y + 90}{8} \Rightarrow 104 = x + y + 90 \Rightarrow x + y = 14 \Rightarrow y = 14 - x$$

The problem with the calculator is that the feature that calculates the variance actually calculates the unbiased estimate of the variance; therefore, we need to multiply by $\frac{n-1}{n}$, which in our case is $\frac{7}{8}$.



Since $x < y$, we can discard the second solution; therefore, $x = 4$ and $y = 14 - 4 = 10$.

Chapter 12

Practice questions

1 **a** Since the events are independent: $P(A \cap B) = P(A) \times P(B) \Rightarrow 0.18 = k \times (k + 0.3) \Rightarrow k^2 + 0.3k - 0.18 = 0$
$(k + 0.6)(k - 0.3) = 0 \Rightarrow k = -0.6$ or $k = 0.3$

Note: Algebraically we get two solutions, but only 0.3 can be a probability value since probability cannot be negative.

b Using the addition formula: $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.6 - 0.18 = 0.72$

c Since the events are independent, the complementary events are independent too.
$P(A'|B') = P(A') \Rightarrow P(A'|B') = 1 - 0.3 = 0.7$

2 **a** Since the tests are taken independently, we multiply the probabilities: $P(A) = 0.02 \cdot 0.02 = 0.0004$

b Using the complementary event: $P(B) = 1 - P(A) = 1 - 0.0004 = 0.9996$

c $P(C) = 0.02 \cdot 0.02 = 0.0004$

Note: The probabilities in parts **a** and **b** are equal because the events are the same.

3 Since they work independently of each other, we need to multiply the probabilities and then use the complementary event.

$P(A') = 1 - P(A) = 1 - 0.002 \times 0.01 = 0.999\,98$

4 **a** **i** Using the addition formula: $P(S \cup F) = P(S) + P(F) - P(S \cap F) = \frac{120}{200} + \frac{60}{200} - \frac{10}{200} = \frac{170}{200} = \frac{17}{20}$

ii Either but not both means that we need to exclude the intersection from the union, so:

$P(S \cup F) - P(S \cap F) = \frac{170}{200} - \frac{10}{200} = \frac{160}{200} = \frac{4}{5}$

iii Does not take any French or Spanish is the complementary event of the union, so:

$P((S \cup F)') = 1 - P(S \cup F) = 1 - \frac{17}{20} = \frac{3}{20}$

b Using the conditional probability formula: $P(F|S) = \frac{P(F \cap S)}{P(S)} = \frac{\frac{1\cancel{0}}{\cancel{200}}}{\frac{12\cancel{0}}{\cancel{200}}} = \frac{1}{12}$

5 It would be a good idea to find the total sums first. There are 126, 84 and 160 disks produced after one run on machines I, II and III respectively. There are a total of 20 defective and 350 non-defective disks. That means there are 370 disks produced in total.

a **i** $P(A_i) = \frac{126}{370} = \frac{63}{185}$

ii $P(A_{ii}) = \frac{4}{370} = \frac{2}{185}$

iii We need to use the addition formula: $P(A_{iii}) = \frac{126 + 350 - 120}{370} = \frac{356}{370} = \frac{178}{185}$.

iv Since this is a conditional probability, our sample space is defective disks and the favourable outcomes are those that are produced by machine I, so $P(A_{iv}) = \frac{6}{20} = \frac{3}{10}$.

b If the quality is independent of the machine, then we can use the multiplication law:

$$\mathrm{P}(M_1)\times\mathrm{P}(D)=\frac{\cancel{126}\,63}{\cancel{370}\,185}\times\frac{\cancel{20}\,1}{\cancel{200}\,10}=\frac{63}{1850}\neq\frac{6}{370}=\mathrm{P}(M_1\cap D)$$. So, the events are dependent.

6 **a** There are 126 envelopes which satisfy our wish, so: $\mathrm{P}(A)=\frac{126}{200}=\frac{63}{100}$.

b There are 68 red envelopes without a prize, so: $\mathrm{P}(B)=\frac{68}{70}=\frac{34}{35}$.

7 **a** $\mathrm{P}(A|B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}\Rightarrow 0.3=\frac{0.18}{\mathrm{P}(B)}\Rightarrow \mathrm{P}(B)=\frac{0.18}{0.3}=0.6$

b The events are independent since $\mathrm{P}(B)=0.6=\mathrm{P}(B|A)$.

c Given that A and B are independent, then A' and B are independent too, so:
$\mathrm{P}(B\cap A')=\mathrm{P}(B)\times\mathrm{P}(A')=0.6\times(1-0.3)=0.42$.

8 **a** Since we know that there are 74 students who took the test, the number of boys who failed is $74-(32+16+12)=14$. There are 6 girls who are too young to take the test and, since there are 10 students altogether that are too young to take the test, the number of boys who are too young is $10-6=4$. Since the total numbers of boys and girls are 70 and 50 respectively, we calculate all of those who were training but did not take the test as: $70-(32+14+4)=20$, and $50-(16+12+6)=16$.

	Boys	**Girls**
Passed the ski test	32	16
Failed the ski test	**14**	12
Training, but did not take the test yet	**20**	**16**
Too young to take the test	**4**	**6**

b **i** $\mathrm{P}(B_i)=\frac{74}{120}=\frac{37}{60}$

ii $\mathrm{P}(B_{ii})=\frac{16+12}{50}=\frac{28}{50}=\frac{14}{25}$

iii These two events are independent so we multiply the probabilities:

$$\mathrm{P}(B_{iii})=\frac{\cancel{32}\,16}{\cancel{70}\,35}\cdot\frac{\cancel{16}\,8}{\cancel{50}\,25}=\frac{128}{875}$$

9 **a** $\mathrm{P}(A\cap B)=\mathrm{P}(A|B)\times\mathrm{P}(B)=\frac{1}{4}\times\frac{3}{8}=\frac{3}{32}$

b $\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B)=\mathrm{P}(A)+\mathrm{P}(B)-2\times\mathrm{P}(A\cap B)=\frac{9}{16}+\frac{3}{8}-\cancel{2}\times\frac{3}{\cancel{32}\,16}=\frac{12}{16}=\frac{3}{4}$

c $\mathrm{P}((A\cup B)')=1-\mathrm{P}(A\cup B)=1-\frac{27}{32}=\frac{5}{32}$

10 **a** Probability that she will miss both serves is: $\mathrm{P}(A)=0.4\cdot 0.05=0.02=2\%$.

b To win a point she will make the first serve and win the point or she will miss the first serve, make the second serve and win the point, so: $\mathrm{P}(B)=0.6\cdot 0.75+0.4\cdot 0.95\cdot 0.5=0.64=64\%$.

11 **a** $\mathrm{P}(A\cap B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cup B)=0.6+0.8-1=0.4$

b $\mathrm{P}(A'\cup B')=\mathrm{P}((A\cap B)')=1-\mathrm{P}(A\cap B)=1-0.4=0.6$

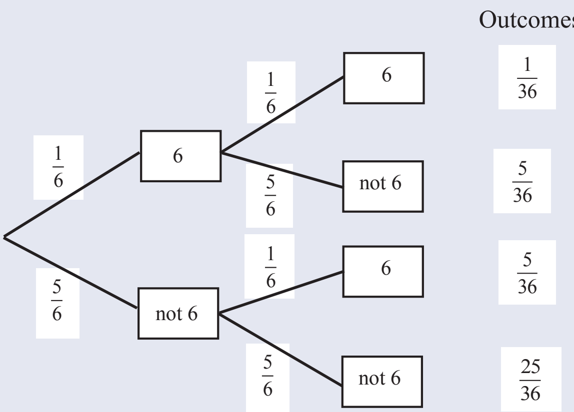
12

	Boys	Girls	Total
Television	13	25	38
Sport	33	29	62
Total	46	54	100

a $\mathrm{P}(A)=\frac{38}{100}=\frac{19}{50}$

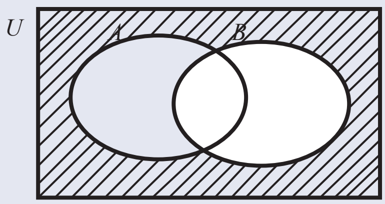
b $\mathrm{P}(B)=\frac{13}{46}$

13 a



b Using the complementary event of not getting a 6: $\mathrm{P}(B)=1-\mathrm{P}(B')=1-\frac{25}{36}=\frac{11}{36}$

14 a



b i Since $n(A\cup B)=n(U)-n((A\cup B)')=36-21=15$, then $n(A\cap B)=n(A)+n(B)-n(A\cup B)=11+6-15=2$.

ii $\mathrm{P}(A\cap B)=\frac{2}{36}=\frac{1}{18}$

c Events A and B are not mutually exclusive since there are two elements in the intersection of the two sets: $A\cap B\neq\varnothing$.

15 a There are 90 females, so there are 110 males. If 60 were unemployed, then 140 were employed. If 20 males were unemployed, then 40 females were unemployed, and so on.

	Males	Females	Totals
Unemployed	20	40	60
Employed	90	50	140
Totals	110	90	200

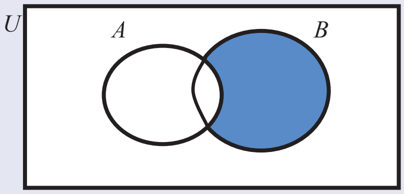
b i $\mathrm{P}(B_i)=\frac{40}{200}=\frac{1}{5}$

ii $\mathrm{P}(B_{ii})=\frac{90}{140}=\frac{9}{14}$

16 There are three possible combinations of different colours and each can occur twice. In total, there are 26 balls in the bag.

$$P(A)=\cancel{2}\cdot\frac{\overbrace{10}^{\text{red}}}{\cancel{26}\,13}\cdot\frac{\overbrace{\cancel{10}\,2}^{\text{green}}}{\cancel{25}\,5}+\cancel{2}\cdot\frac{\overbrace{\cancel{10}\,2}^{\text{red}}}{\cancel{26}\,13}\cdot\frac{\overbrace{6}^{\text{white}}}{\cancel{25}\,5}+\cancel{2}\cdot\frac{\overbrace{6}^{\text{white}}}{\cancel{26}\,13}\cdot\frac{\overbrace{\cancel{10}\,2}^{\text{green}}}{\cancel{25}\,5}=\frac{20+12+12}{65}=\frac{44}{65}$$

17 a



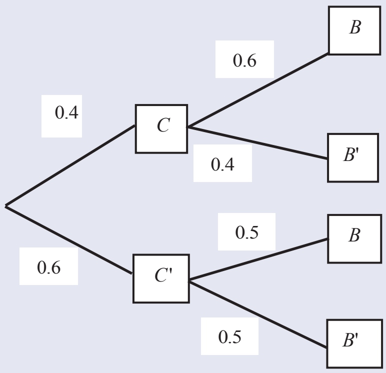
b Firstly, we need to find the number of elements in the intersection:

$n(A\cap B)=n(A)+n(B)-n(A\cup B)=30+50-65=15$

So, $n(B\cap A')=n(B)-n(A\cap B)=50-15=35$.

c $P(B\cap A')=\frac{35}{100}=\frac{7}{20}$

18 a



b The student can take chemistry and biology or not take chemistry but take biology. So, $P(B)=0.4\cdot 0.6+0.6\cdot 0.5=0.54$.

c This is a conditional probability where the favourable event is when a student takes both subjects; therefore,

$P(C|B)=\frac{0.4\cdot 0.6}{0.54}=\frac{4}{9}$.

19 a We will reuse the probability distribution table for the sum from Exercise 10.4 question 6:

S	2	3	4	5	6	7	8	9	10	11	12
P(S)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$P(S<8)=P(S\leqslant 7)=\frac{21}{36}=\frac{7}{12}$

b There are 11 possible pairs when at least one die shows a 3: six pairs with 3 showing on the first die and six pairs with 3 showing on the second die, but the pair (3, 3) should only be counted once. So, $P(B)=\frac{11}{36}$.

c Now, the event from **a** becomes a sample space and we need to find the favourable pairs (out of 21 pairs) found in **b**. The pairs that satisfy the condition are (1, 3), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4) and (4, 3).

$P(B|S\leqslant 7)=\frac{7}{21}=\frac{1}{3}$

20 a $P(A\cap B)=P(A)+P(B)-P(A\cup B)=\frac{3}{11}+\frac{4}{11}-\frac{6}{11}=\frac{1}{11}$

b $P(A\cap B)=P(A)\times P(B)=\frac{3}{11}\times\frac{4}{11}=\frac{12}{121}$

21 a Since the probability of A didn't change when B occurred, $P(A|B)=P(A)$; therefore, the events are independent (I).

b If $P(A\cap B)=0\Rightarrow A\cap B=\varnothing$; therefore, the events are mutually exclusive (M).

c Given that $P(A\cap B)=P(A)\Rightarrow A\cap B=A\Rightarrow A\subseteq B$, neither (N).

22 a $n(E \cup H) = 88 - n((E \cup H)') = 88 - 39 = 49$

Since we know how many study each subject, we can find the intersection:

$b = n(E \cap H) = n(E) + n(H) - n(E \cup H) = 32 + 28 - 49 = 11$

$a = n(E) - n(E \cap H) = 32 - 11 = 21$

$c = n(H) - n(E \cap H) = 28 - 11 = 17$

b Simply by looking at the Venn diagram, since we now have the values of a, b and c:

i $\mathrm{P}(E \cap H) = \dfrac{11}{88} = \dfrac{1}{8}$

ii $\mathrm{P}(E \setminus (E \cap H)) = \dfrac{32 - 11}{88} = \dfrac{21}{88}$

Note: This can be directly read off the diagram.

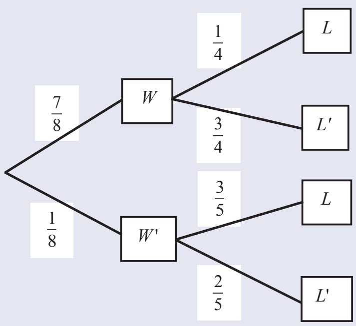
c i There are 56 students who don't study economics and, since a group of three students is selected, we obtain: $\mathrm{P}(E') = \dfrac{56}{88} \cdot \dfrac{55}{87} \cdot \dfrac{54}{86} = \dfrac{315}{1247}$.

ii To find the probability that at least one student studies economics, we are going to use the complementary event (no student studies economics): $1 - \mathrm{P}(E') = 1 - \dfrac{315}{1247} = \dfrac{932}{1247}$.

23 $\mathrm{P}(SC) = \underbrace{\dfrac{7}{12}}_{\text{red}} \cdot \underbrace{\dfrac{6}{11}}_{\text{red}} + \underbrace{\dfrac{5}{12}}_{\text{yellow}} \cdot \underbrace{\dfrac{4}{11}}_{\text{yellow}} = \dfrac{42 + 20}{12 \cdot 11} = \dfrac{\cancel{62}\,31}{\cancel{12}\,6 \cdot 11} = \dfrac{31}{66}$

Note: The tins are chosen without replacement so once a tin is chosen there are 11 tins remaining.

24 a



b $\mathrm{P}(L) = \dfrac{7}{8} \cdot \dfrac{1}{4} + \dfrac{1}{8} \cdot \dfrac{3}{5} = \dfrac{7}{32} + \dfrac{3}{40} = \dfrac{35 + 12}{160} = \dfrac{47}{160}$

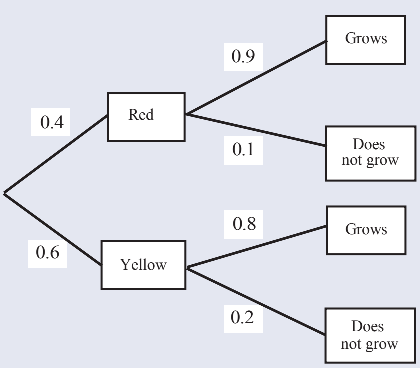
c $\mathrm{P}(W | L) = \dfrac{\mathrm{P}(W \cap L)}{\mathrm{P}(L)} = \dfrac{\frac{7}{\cancel{32}}}{\frac{47}{\cancel{160}\,5}} = \dfrac{35}{47}$

25 a $\mathrm{P}(B) = \dfrac{120^\circ}{360^\circ} = \dfrac{1}{3}$

b $\mathrm{P}(S) = \dfrac{90^\circ + 120^\circ}{360^\circ} = \dfrac{210^\circ}{360^\circ} = \dfrac{7}{12}$

c $\mathrm{P}(A | S) = \dfrac{90^\circ}{210^\circ} = \dfrac{3}{7}$

26 a



b i $\mathrm{P}(R \cap G) = 0.4 \times 0.9 = 0.36$

ii $\mathrm{P}(G) = \underbrace{0.4}_{\text{red}} \cdot \underbrace{0.9}_{\text{grows}} + \underbrace{0.6}_{\text{yellow}} \cdot \underbrace{0.8}_{\text{grows}} = 0.36 + 0.48 = 0.84$

iii $\mathrm{P}(R | G) = \dfrac{\mathrm{P}(R \cap G)}{\mathrm{P}(G)} = \dfrac{0.36}{0.84} = \dfrac{3}{7} \approx 0.429$

27 a There are six favourable outcomes, that is, the same number appears on both dice. There are 36 possible outcomes in total, so: $P(E) = \frac{6}{36} = \frac{1}{6}$.

b There are three possible pairs for a sum of 10: (4, 6), (5, 5) and (6, 4). Therefore, $P(F) = \frac{3}{36} = \frac{1}{12}$.

c To find the probability of the union, we need to apply the addition formula. Note that the events E and F have one common pair, that is, (5, 5).

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{6} + \frac{1}{12} - \frac{1}{36} = \frac{6+3-1}{36} = \frac{8}{36} = \frac{2}{9}$$

28 a i $P(A) = \frac{80}{210} = \frac{8}{21}$

ii $P(A \cap B) = \frac{35}{210} = \frac{1}{6}$

iii To confirm whether the events are independent we firstly find: $P(B) = \frac{100}{210} = \frac{10}{21}$. Now we look at the product of the probabilities: $P(A) \times P(B) = \frac{8}{21} \times \frac{10}{21} = \frac{80}{441} \neq \frac{1}{6} = P(A \cap B)$. Hence, they are dependent.

b $P(Y_1 | H) = \frac{50}{85} = \frac{10}{17}$

c We can select a student from year 1 and a student from year 2, or vice versa; therefore,

$$P(C) = 2 \cdot \frac{11\cancel{0}}{21\cancel{0}} \cdot \frac{100}{\cancel{209}\,19} = \frac{200}{399}.$$

29 Let G be the event that a green ball is chosen and B a blue ball is chosen. If a blue and a green ball have to be selected in any order, we can say that the chosen balls can be blue **and** green **or** green **and** blue.

$$P(BG) + P(GB) = P(B) \cdot P(G) + P(G) \cdot P(B) = \frac{\cancel{3}}{\cancel{9}\,3} \cdot \frac{\cancel{4}}{\cancel{8}\,2} + \frac{\cancel{4}}{\cancel{9}\,3} \cdot \frac{\cancel{3}}{\cancel{8}\,2} = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

30 Let S be the event that the student speaks Spanish as his/her first language, and A that the student is Argentinean.

$$P(S|A) = \frac{P(S) \cdot P(A|S)}{P(A)} = \frac{\frac{15}{21} \cdot \frac{12}{15}}{\frac{15}{21} \cdot \frac{12}{15} + \frac{6}{21} \cdot \frac{3}{6}} = \frac{\frac{4}{7}}{\frac{5}{7}} = \frac{4}{5}$$

31 Let D be the event that a patient has the disease, and T that the patient tests positive.

$$P(D|T) = \frac{P(D) \times P(T|D)}{P(T)} = \frac{0.0001 \times 0.99}{0.0001 \times 0.99 + 0.9999 \times 0.05} \approx 0.00198$$

32 Let H be the event that team A will play against the higher-ranked team, and W that team A win the game.

$$P(W) = P(H) \times P(W|H) + P(H') \times P(W|H') = \frac{3}{9} \times 0.4 + \frac{6}{9} \times 0.75 = \frac{19}{30} \approx 0.633$$

33 $P(A \cap B) + P(A \cap B') = P(A) \Rightarrow P(A) = 0.3 + 0.3 = 0.6$. Since the events are independent, we can use the multiplication law.

$$P(A \cap B) = P(A) \times P(B) \Rightarrow P(B) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.6} = 0.5$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.5 - 0.3 = 0.8$$

34 Let R be the event that it is raining, and L that the girl is late.

$$P(R|L) = \frac{P(R) \cdot P(L|R)}{P(L)} = \frac{\frac{1}{4} \cdot \frac{2}{3}}{\frac{1}{4} \cdot \frac{2}{3} + \frac{3}{4} \cdot \frac{1}{5}} = \frac{\frac{1}{\cancel{6}}}{\frac{19}{\cancel{60}\,10}} = \frac{10}{19}$$

35 a $\mathrm{P}(Y \cap X) = \mathrm{P}(X) \times \mathrm{P}(Y|X) = \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$

$\mathrm{P}(Y \cap X') = \mathrm{P}(X') \times \mathrm{P}(Y|X') = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

$\mathrm{P}(Y) = \mathrm{P}(Y \cap X) + \mathrm{P}(Y \cap X') = \frac{1}{6} + \frac{1}{12} = \frac{1}{4} \Rightarrow \mathrm{P}(Y') = 1 - \mathrm{P}(Y) = 1 - \frac{1}{4} = \frac{3}{4}$

b $\mathrm{P}(X' \cup Y') = \mathrm{P}((X \cap Y)') = 1 - \mathrm{P}(X \cap Y) = 1 - \frac{1}{6} = \frac{5}{6}$

36 Let F be the event that the umbrella is left in the first shop, and S that the umbrella is left in the second shop.

$$\mathrm{P}(S|F \cup S) = \frac{\frac{2}{3} \times \frac{1}{3}}{\frac{1}{3} + \frac{2}{3} \times \frac{1}{3}} = \frac{\frac{2}{\cancel{9}}}{\frac{5}{\cancel{9}}} = \frac{2}{5}$$

37 a i $\mathrm{P}(A_1) = \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}$

ii $\mathrm{P}(A_2) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{25}{216}$

iii $\mathrm{P}(A_n) = \left(\frac{5}{6} \times \frac{5}{6}\right)^{n-1} \times \frac{1}{6} = \frac{1}{6}\left(\frac{25}{36}\right)^{n-1} = \frac{1}{6}\left(\frac{5}{6}\right)^{2n-2}$

b We have an infinite geometric series:

$$p = \frac{1}{6} + \frac{25}{36} \times \frac{1}{6} + \left(\frac{25}{36}\right)^2 \frac{1}{6} + \left(\frac{25}{36}\right)^3 \frac{1}{6} + \ldots = \frac{1}{6} + \frac{25}{36}\overbrace{\left(\frac{1}{6} + \frac{25}{36} \times \frac{1}{6} + \left(\frac{25}{36}\right)^2 \frac{1}{6} + \left(\frac{25}{36}\right)^3 \frac{1}{6} + \ldots\right)}^{p} \Rightarrow p = \frac{1}{6} + \frac{25}{36} \times p$$

c Firstly, we will calculate the probability that Ann wins the game:

$$p = \frac{1}{6} + \frac{25}{36} \times p \Rightarrow p - \frac{25}{36} p = \frac{1}{6} \Rightarrow \frac{11}{36} p = \frac{1}{6} \Rightarrow p = \frac{1}{\cancel{6}} \times \frac{\cancel{36}\,6}{11} \Rightarrow p = \frac{6}{11}$$

The probability that Bridget wins the game is complementary to the above event, so: $\mathrm{P}(\text{Bridget wins}) = 1 - \frac{6}{11} = \frac{5}{11}$

d If Ann wins more games than Bridget, that means that she has to win 4, 5 or 6 times. We also need to find the number of sequences with that number of wins. For example, if Ann wins four out of six games played, she can do that in '6 choose 4' different ways (we are using the binomial coefficients).

$$\mathrm{P}(D) = \binom{6}{4}\overbrace{\left(\frac{6}{11}\right)^4}^{\text{Ann wins 4 times}}\overbrace{\left(\frac{5}{11}\right)^2}^{\text{Bridget wins twice}} + \binom{6}{5}\overbrace{\left(\frac{6}{11}\right)^5}^{\text{Ann wins 5 times}}\overbrace{\left(\frac{5}{11}\right)}^{\text{Bridget wins once}} + \overbrace{\left(\frac{6}{11}\right)^6}^{\text{Ann wins 6 times}} \approx 0.432$$

38 a If the first two selected apples are green, then 22 red apples and 1 green apple remain in the box. Therefore, the probability that the next apple will be red is $\frac{22}{23} \approx 0.957$.

b There are three different selections that will give the result of exactly two red apples: RRG, RGR or GRR. So, the probability is calculated as follows: $\mathrm{P}(B) = 3 \times \frac{\cancel{22}\,11}{25} \times \frac{21}{\cancel{24}\,4} \times \frac{\cancel{3}}{23} = \frac{693}{2300} \approx 0.301$

39 Let R be the event that it rains during the day, and T that the daily maximum temperature exceeds 25 °C.

$\mathrm{P}(T) = \mathrm{P}(R) \cdot \mathrm{P}(T|R) + \mathrm{P}(R') \cdot \mathrm{P}(T|R') = 0.2 \cdot 0.3 + 0.8 \cdot 0.6 = 0.54$

$\mathrm{P}(R|T) = \frac{\mathrm{P}(R) \times \mathrm{P}(T|R)}{\mathrm{P}(T)} = \frac{0.06}{0.54} = \frac{1}{9} \approx 0.111$

40 a The number of integers up to the number n that are divisible by p is given by $\left\lfloor \frac{n}{p} \right\rfloor$, which denotes the greatest integer of the number $\frac{n}{p}$. The number of multiples of 4 from the first 1000 numbers is 250; therefore, the probability that we select one such number is: $\mathrm{P}(A) = \frac{250}{1000} = \frac{1}{4}$.

b The number of integers that are divisible by both 4 and 6 is actually the number of integers that are divisible by their least common multiple, that is, 12: $\left\lfloor \frac{1000}{12} \right\rfloor = \lfloor 83.3333... \rfloor = 83$. The probability that we select one such number is: $\mathrm{P}(B) = \frac{83}{1000} = 0.083$.

41 a Given that the events A and B are independent, we can use the multiplication law; therefore, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$. In combination with the addition law, we can calculate:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A) \times \mathrm{P}(B)$$

$$0.88 = 0.4 + \mathrm{P}(B) - 0.4 \times \mathrm{P}(B) \Rightarrow 0.48 = 0.6\mathrm{P}(B) \Rightarrow \mathrm{P}(B) = \frac{0.48}{0.6} = 0.8$$

b This part can be calculated in two different ways:

Method I: $\mathrm{P}(A \cup B) - \mathrm{P}(A \cap B) = 0.88 - 0.4 \times 0.8 = 0.88 - 0.32 = 0.56$

Method II: $\mathrm{P}(A \cap B') + \mathrm{P}(A' \cap B) = \mathrm{P}(A) \times \mathrm{P}(B') + \mathrm{P}(A') \times \mathrm{P}(B) = 0.4 \times 0.2 - 0.6 \times 0.8 = 0.56$

42 Let M be the event that the chosen day is Monday, and T that Robert catches the 08:00 train.

a $\mathrm{P}(T) = \mathrm{P}(M) \cdot \mathrm{P}(T|M) + \mathrm{P}(M') \cdot \mathrm{P}(T|M') = 0.2 \cdot 0.66 + 0.8 \cdot 0.75 = 0.732$

b $\mathrm{P}(M|T) = \dfrac{\mathrm{P}(M) \times \mathrm{P}(T|M)}{\mathrm{P}(T)} = \dfrac{0.132}{0.732} = \dfrac{11}{61} \approx 0.180$

43 a $\mathrm{P}(A) = \frac{4}{6} = \frac{2}{3}$

b $\mathrm{P}(B) = \frac{2}{6} \cdot \frac{4}{6} = \frac{2}{9}$

c We are going to have an infinite geometric series:

$$\mathrm{P}(C) = \frac{2}{3} + \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} + \left(\frac{1}{3} \times \frac{1}{3}\right)^2 \times \frac{2}{3} + \left(\frac{1}{3} \times \frac{1}{3}\right)^3 \times \frac{2}{3} + ... = \frac{2}{3}\left(1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{9}\right)^3 + ...\right) = \frac{2}{3} \times \frac{1}{1 - \frac{1}{9}} = \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{9}\,3}{\cancel{8}\,4} = \frac{3}{4}$$

44 a $\mathrm{P}(A) = \dfrac{\cancel{2}}{5} \cdot \dfrac{1}{\cancel{4}\,2} = \dfrac{1}{10}$

b $\mathrm{P}(B) = \dfrac{4}{n+4} \times \dfrac{3}{n+3} = \dfrac{12}{(n+4)(n+3)} \Rightarrow \dfrac{12}{(n+4)(n+3)} = \dfrac{2}{15} \Rightarrow (n+4)(n+3) = 90 \Rightarrow$

$n^2 + 7n - 78 = 0 \Rightarrow (n+13)(n-6) = 0 \Rightarrow \cancel{n = -13}$ or $n = 6$. We can discard the negative solution since the number of balls cannot be negative.

c $\mathrm{P}(C) = \underbrace{\dfrac{1}{3}}_{\text{Bag A}} \cdot \underbrace{\dfrac{1}{10}}_{\text{Two red}} + \underbrace{\dfrac{2}{3}}_{\text{Bag B}} \cdot \underbrace{\dfrac{2}{15}}_{\text{Two red}} = \dfrac{3+8}{90} = \dfrac{11}{90}$

d Let C be the event that two red balls are drawn, and X that the balls are drawn from bag A.

$$\mathrm{P}(X|C) = \frac{\mathrm{P}(X) \cdot \mathrm{P}(C|X)}{\mathrm{P}(C)} = \frac{\frac{1}{\cancel{30}}}{\frac{11}{\cancel{90}\,3}} = \frac{3}{11}$$

45 The first fact stated in the problem gives us the following: $(A \cup B)' = \varnothing \Rightarrow \mathrm{P}(A \cup B) = 1$.

$$P(A \cup B) = P(A) + P(B) - \mathrm{P}(A \cap B) \Rightarrow 1 = \frac{6}{7} + \mathrm{P}(B) - \mathrm{P}(A \cap B) \Rightarrow \mathrm{P}(A \cap B) = \mathrm{P}(B) - \frac{1}{7}$$

$$\mathrm{P}(A'|B) = \frac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)} \Rightarrow \mathrm{P}(A' \cap B) = \frac{1}{3}\mathrm{P}(B)$$

$$\mathrm{P}(B) = \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B) = \mathrm{P}(B) - \frac{1}{7} + \frac{1}{3}\mathrm{P}(B) \Rightarrow \frac{1}{3}\mathrm{P}(B) = \frac{1}{7} \Rightarrow \mathrm{P}(B) = \frac{3}{7}$$

Chapter 13

Practice questions

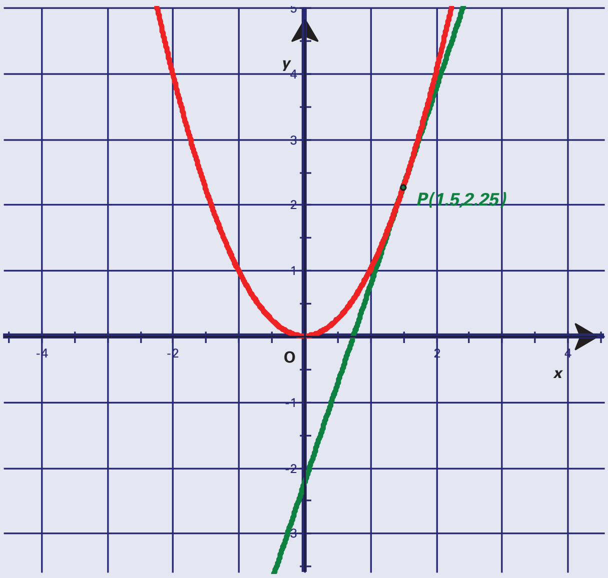
1 For $f(x) = x^2$ we have:

a $f'(x) = 2x \Rightarrow f'(1.5) = 3$ is the slope

b $f(1.5) = 1.5^2 = 2.25 \Rightarrow P(1.5, 2.25)$

$y - 2.25 = 3(x - 1.5) \Rightarrow y = 3x - 2.25$

c



d $y = 0 \Rightarrow 0 = 3x - 2.25 \Rightarrow x = 0.75 \Rightarrow Q(0.75, 0)$

$x = 0 \Rightarrow y = -2.25 \Rightarrow R(0, -2.25)$

e For $P(1.5, 2.25)$ and $R(0, -2.25)$, the midpoint has coordinates:

$$x_{midpt} = \frac{1.5 + 0}{2} = 0.75 = x_Q,\ y_{midpt} = \frac{2.25 - 2.25}{2} = 0 = y_Q$$

Hence, Q is the midpoint of PR.

f $f'(a) = 2a \Rightarrow y - a^2 = 2a(x - a) \Rightarrow y = 2ax - a^2$

g $y = 0 \Rightarrow 0 = 2ax - a^2 \Rightarrow x = \frac{a}{2} \Rightarrow T\left(\frac{a}{2}, 0\right)$

$x = 0 \Rightarrow y = -a^2 \Rightarrow U(0, -a^2)$

h For $S(a, a^2)$ and $U(0, -a^2)$, the coordinates of the midpoint are:

$$x_{MSU} = \frac{a + 0}{2} = \frac{a}{2} = x_T,\ y_{MSU} = \frac{a^2 - a^2}{2} = 0 = y_T$$

Hence, T is the midpoint of SU.

2 For $y = Ax + B + \frac{C}{x}, x \in \mathbb{R}, x \neq 0$:

$f(1) = A + B + C = 4$

$f(-1) = -A + B - C = 0$

$f'(x) = A - \frac{C}{x^2}$

$f'(1) = A - C = 0, f'(-1) = A - C = 0$, since (1, 4) and (−1, 0) are points of extrema.

The system of equations is:

$$\begin{cases} A + B + C = 4 \\ -A + B - C = 0 \\ A - C = 0 \end{cases} \Rightarrow A = 1, B = 2, C = 1$$

3 **a** $\frac{d}{dx}\left(x^2(2 - 3x^3)\right) = \frac{d}{dx}\left(2x^2 - 3x^5\right) = 4x - 15x^4$

b $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -x^{-2} = -\frac{1}{x^2}$

4 For $f(x) = \frac{8}{x} + 2x, x > 0$, we have:

a $f'(x) = -\frac{8}{x^2} + 2 = 0 \Rightarrow -8 + 2x^2 = 0 \Rightarrow x = 2, \cancel{x = -2}$

$f'(1) = -8 + 2 = -6 < 0 \Rightarrow f(x) \searrow$ on (0, 2)

$f'(3) = -\frac{8}{9} + 2 = \frac{10}{9} > 0 \Rightarrow f(x) \nearrow$ on $(0, \infty)$

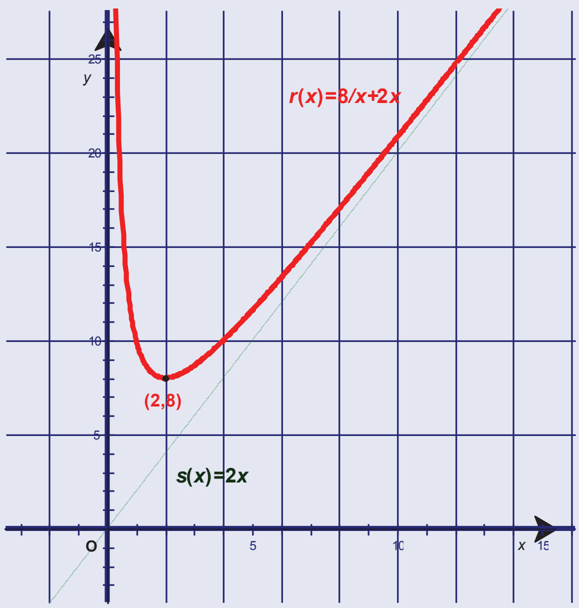
$f(2) = \frac{8}{2} + 2 \cdot 2 = 8$

Hence, point (2, 8) is the turning point (absolute minimum).

b For $x = 0$ the function is not defined, so $x = 0$ is the vertical asymptote.

For $x \to \infty, \frac{8}{x} \to 0 \Rightarrow f(x) \to 2x$

Oblique asymptote is $y = 2x$.



5 For $y = 4x^2 + \frac{1}{x}$ we have: $\frac{dy}{dx} = 8x - \frac{1}{x^2} = 0 \Rightarrow 8x^3 = 1 \Rightarrow x = \frac{1}{2}$

For $x = \frac{1}{2}$: $y = 4 \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{\frac{1}{2}} = 3$.

So, the stationary point is $\left(\frac{1}{2}, 3\right)$.

6 For $y = ax^3 - 2x^2 - x + 7$ we have: $\dfrac{dy}{dx} = 3ax^2 - 4x - 1$

At $x = 2 \Rightarrow 3a \cdot 4 - 4 \cdot 2 - 1 = 3 \Rightarrow a = 1$

7 Given $f(2) = 3$ and $f'(2) = 5$:

a The equation of the tangent at (2, 3) is: $y - 3 = 5(x - 2) \Rightarrow y = 5x - 7$.

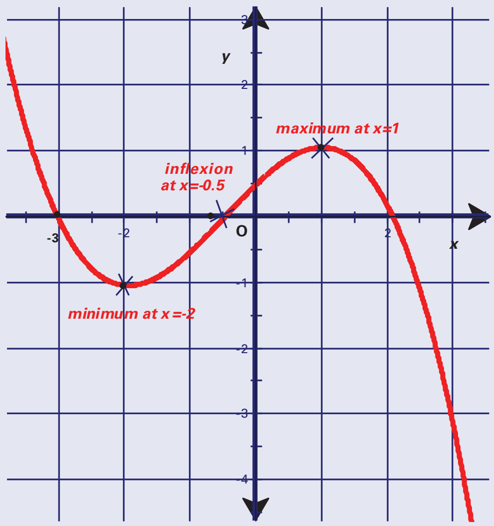
b The normal has a slope of $-\dfrac{1}{5}$, so the equation of the normal at (2, 3) is: $y - 3 = -\dfrac{1}{5}(x - 2) \Rightarrow y = -\dfrac{1}{5}x + \dfrac{17}{5}$.

8 **a** $g(x)$ has a maximum at $x = 1$ because $g'(x)$ is positive before it, and negative after it.

b The value of $g(x)$ is decreasing when $g'(x) < 0$, i.e. $x \in (-3, -2) \cup (1, 3)$.

c $g(x)$ has a point of inflexion at $x = -\dfrac{1}{2}$ because $g''\left(-\dfrac{1}{2}\right) = 0$.

d



9 For $f(x) = x^2 - 3bx + (c + 2)$ we have:

$f(1) = 1 - 3b + c + 2 = 0 \Rightarrow 3b - c = 3$

$f'(x) = 2x - 3b$

$f'(3) = 6 - 3b = 0$

The system of equations is:

$$\begin{cases} 3b - c = 3 \\ 6 - 3b = 0 \end{cases} \Rightarrow b = 2, c = 3$$

10

Function	**Derivative diagram**	**Reason**
f_1	d	$f(x) \searrow \Rightarrow f' < 0$, then $f(x) \nearrow \Rightarrow f' > 0$
f_2	e	$f(x) \searrow \Rightarrow f' < 0$, $f(x)$ has a point of inflexion at $x = 0 \Rightarrow f'$ extreme at $x = 0$
f_3	b	where $f(x)$ has extremes $f'(x) = 0$
f_4	a	$f(x) \searrow \Rightarrow f' < 0$

11 For $f(x) = 1 + \sin x$:

a The average rate of change is: $\dfrac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} = \dfrac{\left(1 + \sin\frac{\pi}{2}\right) - (1 + \sin 0)}{\frac{\pi}{2}} = \dfrac{1}{\frac{\pi}{2}} = \dfrac{2}{\pi}$

b The instantaneous rate of change: $f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

c $f'(x) = \frac{2}{\pi}, 0 < x < \frac{\pi}{2} \Rightarrow \cos x = \frac{2}{\pi} \Rightarrow x = \arccos\left(\frac{2}{\pi}\right) \approx 0.881$

12 For $y = \frac{3x - 2}{x} = 3 - \frac{2}{x}$ we have:

a **i** The vertical asymptote is $x = 0$ because the function is not defined there.

ii The horizontal asymptote is $y = 3$ because for $x \to \pm\infty : y \to 3$.

b $\frac{dy}{dx} = -2(-1)x^{-2} = \frac{2}{x^2}$

c $\frac{dy}{dx} > 0$ for the whole domain, so y is increasing for $x \in (-\infty, 0) \cup (0, \infty)$.

d $\frac{dy}{dx} \neq 0$ for the whole domain, so there are no stationary points.

13 For $h(x) = 2x^2 - x^4$ we have:

$h'(x) = 4x - 4x^3 = 4x(1 - x^2) = 4x(1 + x)(1 - x)$

$h'(x) = 0 \Rightarrow x = -1, x = 0, x = 1$

$h(-1) = 1, h(0) = 0, h(1) = 1$

Since the function is continuous, three stationary points, (−1, 1), (0, 0), (1, 1), mean that there are four intervals that need to be tested:

$h'(-2) = 4 \cdot (-2) - 4 \cdot (-2)^3 = 24 > 0 \Rightarrow h(x) \nearrow$ on $(-\infty, -1)$

$h'\left(-\frac{1}{2}\right) = 4 \cdot \left(-\frac{1}{2}\right) - 4 \cdot \left(-\frac{1}{2}\right)^3 = -\frac{3}{2} < 0 \Rightarrow h(x) \searrow$ on $(-1, 0)$

$h'\left(\frac{1}{2}\right) = 4 \cdot \left(\frac{1}{2}\right) - 4 \cdot \left(\frac{1}{2}\right)^3 = \frac{3}{2} > 0 \Rightarrow h(x) \nearrow$ on $(0, 1)$

$h'(2) = 4 \cdot 2 - 4 \cdot 2^3 = -24 < 0 \Rightarrow h(x) \searrow$ on $(1, \infty)$

Therefore, the function has maximum points at (−1, 1) and (1, 1), and a minimum point at (0, 0).

14 For the curve $y = x^{\frac{1}{2}} + x^{\frac{1}{3}}$ we have:

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}}$

At $x = 1$: $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

The slope of the normal is: $-\dfrac{1}{\frac{5}{6}} = -\dfrac{6}{5}$.

The equation of the normal at (1, 2) is: $y - 2 = -\frac{6}{5}(x - 1) \Rightarrow y = -\frac{6}{5}x + \frac{16}{5}$.

The normal intersects the axis at:

$$y = 0 \Rightarrow 0 = -\frac{6}{5}x + \frac{16}{5} \Rightarrow x = \frac{8}{3}$$

$$x = 0 \Rightarrow y = \frac{16}{5}$$

So, $a = \frac{8}{3}, b = \frac{16}{5}$.

15 For the displacement function $s(t) = 10t - \frac{1}{2}t^2, t \geqslant 0$:

a The velocity function is $v(t) = s'(t) = 10 - t$.

When $t = 0, v(0) = 10$ m/s.

b $v(t) = 0 \Rightarrow t = 10$ seconds

c $s(10) = 10 \cdot 10 - \frac{1}{2} \cdot 10^2 = 50$ metres

16 For the height function $h = h(t) = 14t - 4.9t^2, t \geqslant 0$:

a Velocity: $v(t) = h'(t) = 14 - 9.8t$
Acceleration: $a(t) = v'(t) = -9.8$

b $h'(t) = 0 \Rightarrow 14 - 9.8t = 0 \Rightarrow t \approx 1.43$ seconds
$h(1.43) = 14 \cdot 1.43 - 4.9 \cdot 1.43^2 \approx 10$ metres

c $v(1.43) = 0$ m/s

$a(1.43) = -9.8$ m/s^2

17 For $y = x^3 + 12x^2 - x - 12$ we have:

$$\frac{dy}{dx} = 3x^2 + 24x - 1, \frac{d^2y}{dx^2} = 6x + 24$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow 6x + 24 = 0 \Rightarrow x = -4$$

For $x = -4$, $y = (-4)^3 + 12(-4)^2 - (-4) - 12 = 120$

The inflexion point is $(-4, 120)$.

18 For $f(x) = 2\cos x - 3$ we have:

a $f'(x) = -2\sin x, f'\left(\frac{\pi}{3}\right) = -2\sin\left(\frac{\pi}{3}\right) = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3}$

$f\left(\frac{\pi}{3}\right) = 2\cos\left(\frac{\pi}{3}\right) - 3 = 2 \cdot \frac{1}{2} - 3 = -2$

Tangent at $\left(\frac{\pi}{3}, -2\right)$: $y + 2 = -\sqrt{3}\left(x - \frac{\pi}{3}\right) \Rightarrow y = -\sqrt{3}x + \frac{\pi\sqrt{3}}{3} - 2$

b Normal at $\left(\frac{\pi}{3}, -2\right)$: $y + 2 = \frac{1}{\sqrt{3}}\left(x - \frac{\pi}{3}\right) \Rightarrow y = \frac{\sqrt{3}}{3}x - \frac{\pi\sqrt{3}}{9} - 2$

19 a Surface area: $S = 2r\pi(r + h) = 54\pi \Rightarrow r^2 + rh = 27 \Rightarrow h = \frac{27 - r^2}{r}$

Volume: $V = V(h) = r^2\pi h = r^2\pi \cdot \frac{27 - r^2}{r} = r\pi(27 - r^2) = \pi(27r - r^3)$

b $V'(h) = \pi(27 - 3r^2) = 0 \Rightarrow (3-r)(3+r) = 0 \Rightarrow r = 3, \cancel{r = -3}$

The volume is a maximum for $r = 3$ cm.

20 For $y = ax^2 + bx + c$:

Passes through $(2, 18) \Rightarrow 4a + 2b + c = 18$

Passes through $(0, 10) \Rightarrow c = 10$

Maximum at $x = 2$: $\dfrac{dy}{dx} = 0$ and $\dfrac{dy}{dx} = 2ax + b \Rightarrow 4a + b = 0$

The system of equations:

$$\begin{cases} 4a + 2b + c = 18 \\ 4a + b = 0 \\ c = 10 \end{cases} \Rightarrow \begin{cases} 4a + 2b = 8 \\ 4a + b = 0 \end{cases} \Rightarrow b = 8, a = -2$$

The function is $y = -2x^2 + 8x + 10$.

21 For the function $f(x) = \dfrac{1}{2}x^2 - 5x + 3$:

a $f(-2) = \dfrac{1}{2}(-2)^2 - 5 \cdot (-2) + 3 = 15, f'(x) = x - 5, f'(-2) = -2 - 5 = -7$

Tangent at $(-2, 15)$: $y - 15 = -7(x + 2) \Rightarrow y = -7x + 1$

b Normal at $(-2, 15)$: $y - 15 = \dfrac{1}{7}(x + 2) \Rightarrow y = \dfrac{1}{7}x + \dfrac{107}{7}$

22 For $f(x) = x^4 - x^3$ we have:

a $f'(x) = 4x^3 - 3x^2 = 0 \Rightarrow x^2(4x - 3) = 0 \Rightarrow x = 0, x = \dfrac{3}{4}$

$f'(-1) = 4(-1)^3 - 3(-1)^2 = -7 < 0 \Rightarrow f(x) \searrow$ on $(-\infty, 0)$

$f'\left(\dfrac{1}{2}\right) = 4\left(\dfrac{1}{2}\right)^3 - 3\left(\dfrac{1}{2}\right)^2 = -\dfrac{1}{4} < 0 \Rightarrow f(x) \searrow$ on $\left(0, \dfrac{3}{4}\right)$

$f'(1) = 4 - 3 = 1 > 0 \Rightarrow f(x) \nearrow$ on $\left(\dfrac{3}{4}, \infty\right)$

$f\left(\dfrac{3}{4}\right) = \left(\dfrac{3}{4}\right)^4 - \left(\dfrac{3}{4}\right)^3 = -\dfrac{27}{256}$

Absolute minimum at $\left(\dfrac{3}{4}, -\dfrac{27}{256}\right)$.

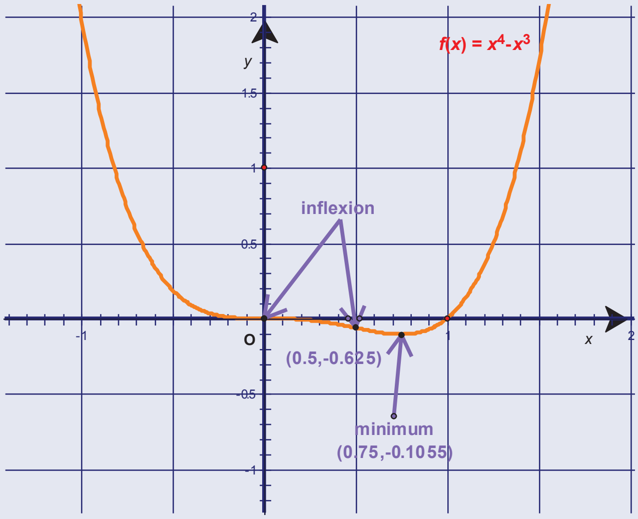
b Domain is $\mathbb{R}$ and the range is $\left(-\dfrac{27}{256}, \infty\right)$.

c $f''(x) = 12x^2 - 6x = 0 \Rightarrow 6x(2x - 1) = 0 \Rightarrow x = 0, x = \dfrac{1}{2}$

$f(0) = 0, f\left(\dfrac{1}{2}\right) = \left(\dfrac{1}{2}\right)^4 - \left(\dfrac{1}{2}\right)^3 = -\dfrac{1}{16}$

The inflexion points are $(0, 0)$ and $\left(\dfrac{1}{2}, -\dfrac{1}{16}\right)$.

d



23 a $\lim_{x\to\infty}\dfrac{2-3x+5x^2}{8-3x^2} = \lim_{x\to\infty}\dfrac{\frac{2-3x+5x^2}{x^2}}{\frac{8-3x^2}{x^2}} = \lim_{x\to\infty}\dfrac{\frac{2}{x^2}-\frac{3}{x}+5}{\frac{8}{x^2}-3} = \dfrac{0-0+5}{0-3} = -\dfrac{5}{3}$

b $\lim_{x\to 0}\dfrac{\sqrt{x+4}-2}{x} = \lim_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}\cdot\dfrac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \lim_{x\to 0}\dfrac{x+4-4}{x(\sqrt{x+4}+2)}$

$= \lim_{x\to 0}\dfrac{\cancel{x}}{\cancel{x}(\sqrt{x+4}+2)} = \lim_{x\to 0}\dfrac{1}{(\sqrt{x+4}+2)} = \dfrac{1}{\sqrt{0+4}+2} = \dfrac{1}{4}$

c $\lim_{x\to 1}\dfrac{x^3-1}{x-1} = \lim_{x\to 1}\dfrac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = \lim_{x\to 1}(x^2+x+1) = 1+1+1 = 3$

d $\lim_{h\to 0}\dfrac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h} = \lim_{h\to 0}\dfrac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h}\cdot\dfrac{\sqrt{(x+h)+2}+\sqrt{x+2}}{\sqrt{(x+h)+2}+\sqrt{x+2}}$

$= \lim_{h\to 0}\dfrac{[(x+h)+2]-[x+2]}{h\left(\sqrt{(x+h)+2}+\sqrt{x+2}\right)} = \lim_{h\to 0}\dfrac{\cancel{h}}{\cancel{h}\left(\sqrt{(x+h)+2}+\sqrt{x+2}\right)} = \dfrac{1}{\sqrt{(x+0)+2}+\sqrt{x+2}} = \dfrac{1}{2\sqrt{x+2}}$

24 a $f(x) = \dfrac{x^2-4x}{\sqrt{x}} = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$

$f'(x) = \dfrac{3}{2}x^{\frac{1}{2}} - 4\cdot\dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{3\sqrt{x}}{2} - \dfrac{2}{\sqrt{x}} = \dfrac{3x-4}{2\sqrt{x}}$

b $f(x) = x^3 - 3\sin x$

$f'(x) = 3x^2 - 3\cos x$

c $f(x) = \dfrac{1}{x} + \dfrac{x}{2} = x^{-1} + \dfrac{x}{2}$

$f'(x) = (-1)x^{-2} + \dfrac{1}{2} = -\dfrac{1}{x^2} + \dfrac{1}{2}$

d $f(x) = \dfrac{7}{3x^{13}} = \dfrac{7}{3}x^{-13}$

$f'(x) = \dfrac{7}{3}\cdot(-13)x^{-14} = -\dfrac{91}{3x^{14}}$

25 For the curve $y = x^3 + x^2 - 9x - 9$, we have $\frac{dy}{dx} = 3x^2 + 2x - 9$.

The slope of the tangent at (p, q) is $3p^2 + 2p - 9$.

As (p, q) is on the curve, $q = p^3 + p^2 - 9p - 9$.

The equation of the tangent at (p, q) is $y - (p^3 + p^2 - 9p - 9) = (3p^2 + 2p - 9)(x - p)$.

The point $(4, -1)$ is on the tangent, so:

$-1 - (p^3 + p^2 - 9p - 9) = (3p^2 + 2p - 9)(4 - p) \Rightarrow -1 - p^3 - p^2 + 9p + 9 = 12p^2 + 8p - 36 - 3p^3 - 2p^2 + 9p \Rightarrow$

$2p^3 - 11p^2 - 8p + 44 = 0 \Rightarrow 2p^3 - 4p^2 - 7p^2 + 14p - 22p + 44 = 0 \Rightarrow$

$2p^2(p - 2) - 7p(p - 2) - 22(p - 2) = 0 \Rightarrow$

$(p - 2)(2p^2 - 7p - 22) = 0 \Rightarrow$

$(p - 2)(p + 2)(2p - 11) = 0 \Rightarrow p_1 = 2, p_2 = -2, p_3 = \frac{11}{2}$

$q_1 = 2^3 + 2^2 - 9 \cdot 2 - 9 = -15$

$q_2 = (-2)^3 + (-2)^2 - 9(-2) - 9 = 5$

$q_3 = \left(\frac{11}{2}\right)^3 + \left(\frac{11}{2}\right)^2 - 9 \cdot \frac{11}{2} - 9 = \frac{1105}{8}$

There are three solutions for point (p, q): $(2, -15), (-2, 5), \left(\frac{11}{2}, \frac{1105}{8}\right)$.

26 Let $\left(a, a^3 + \frac{1}{3}\right)$ be the point at which the normal $y = -\frac{1}{12}x + c, c \geqslant 0$, intersects the curve $y = x^3 + \frac{1}{3}$. The slope function for this curve is $\frac{dy}{dx} = 3x^3$, so the slope of the normal at $x = a$ is $-\frac{1}{3a^2}$. From the equation of the normal, we can see that the slope is $-\frac{1}{12}$, so $-\frac{1}{3a^2} = -\frac{1}{12} \Rightarrow a^2 = 4 \Rightarrow a_1 = 2, a_2 = -2$.

For $a_1 = 2$, the intersection point is $\left(2, \frac{25}{3}\right)$. It is on the normal, so: $\frac{25}{3} = -\frac{1}{12} \cdot 2 + c \Rightarrow c_1 = \frac{17}{2}$.

For $a_2 = -2$, the intersection point is $\left(-2, -\frac{23}{3}\right)$. It is on the normal, so:

$-\frac{23}{3} = -\frac{1}{12} \cdot (-2) + c \Rightarrow c_2 = -\frac{47}{6}$. This solution is negative, so the only solution is $c = \frac{17}{2}$.

27 For the curve $y = \frac{1}{3}x^3 - x$, the slope function is $\frac{dy}{dx} = x^2 - 1$. If the tangent is parallel to the line $y = 3x$, it should have the same slope, so: $x^2 - 1 = 3 \Rightarrow x^2 = 4 \Rightarrow x_1 = 2, x_2 = -2$.

For $x_1 = 2: y_1 = \frac{1}{3} \cdot 2^3 - 2 = \frac{2}{3}$.

For $x_2 = -2: y_2 = \frac{1}{3} \cdot (-2)^3 - (-2) = -\frac{2}{3}$.

There are two points with tangents parallel to the given line: $\left(2, \frac{2}{3}\right)$ and $\left(-2, -\frac{2}{3}\right)$.

28 For the curve $y = x - x^2$, the slope function is $\frac{dy}{dx} = 1 - 2x$.

The slope of the normal at $(1, 0)$ is $-\frac{1}{1-2} = 1$, so the equation of the normal is: $y - 0 = 1 \cdot (x - 1) \Rightarrow y = x - 1$.

To find the intersection points, we have to solve the following system of equations:

$$\begin{cases} y = x - x^2 \\ y = x - 1 \end{cases} \Rightarrow x - x^2 = x - 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

We have been given the intersection point $(1, 0)$, so for $x = -1: y = -1 - (-1)^2 = -2$.

Point $(-1, -2)$ is the other intersection point.

29 For $f(x) = \sqrt{x+2}$ we have:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h}\cdot\frac{\sqrt{(x+h)+2}+\sqrt{x+2}}{\sqrt{(x+h)+2}+\sqrt{x+2}} = \lim_{h\to 0}\frac{[(x+h)+2]-[x+2]}{h\left(\sqrt{(x+h)+2}+\sqrt{x+2}\right)}$$

$$= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{(x+h)+2}+\sqrt{x+2}\right)} = \frac{1}{\sqrt{(x+0)+2}+\sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$$

30 For the position function $s(t) = t^3 - 9t^2 + 24t$:

a The velocity function:

$v(t) = \dfrac{ds}{dt} = 3t^2 - 18t + 24$

$v(t) = 0 \Rightarrow 3t^2 - 18t + 24 = 0 \Rightarrow t^2 - 6t + 8 = 0 \Rightarrow (t-2)(t-4) = 0 \Rightarrow t = 2, t = 4$

$s(2) = 2^3 - 9\cdot 2^2 + 24\cdot 2 = 20$

$s(4) = 4^3 - 9\cdot 4^2 + 24\cdot 4 = 16$

b The acceleration function:

$a(t) = \dfrac{dv}{dt} = 6t - 18$

$a(t) = 0 \Rightarrow 6t - 18 = 0 \Rightarrow t = 3$

$s(3) = 3^3 - 9\cdot 3^2 + 24\cdot 3 = 18$

31 For the displacement function $s(t) = t + \sin t, 0 \leqslant t \leqslant 2\pi$:

a The velocity function is $v(t) = s'(t) = 1 + \cos t$.

Since $-1 \leqslant \cos t \leqslant 1 \Rightarrow v(t) \geqslant 0$, which means that the particle does not change direction.

b For $0 \leqslant t \leqslant \pi$: $\sin t \geqslant 0 \Rightarrow t + \sin t \geqslant 0$

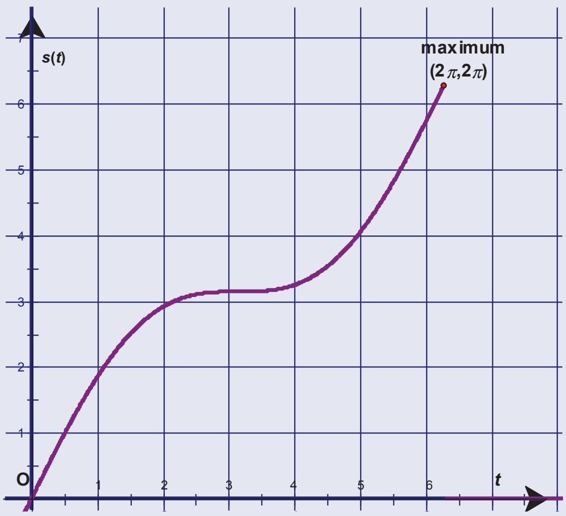
For $\pi \leqslant t \leqslant 2\pi$: $-1 \leqslant \sin t \leqslant 0 \Rightarrow t + \sin t \geqslant \pi - 1 \geqslant 0$

So, $s(t) \geqslant 0$ for $0 \leqslant t \leqslant 2\pi$, which means that the particle is always on the same side of the origin.

c The acceleration function is $a(t) = v'(t) = -\sin t$.

$a(t) = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0, t = \pi, t = 2\pi$

d



32 For the curve $y = ax^3 + bx^2 + cx + d$: $\dfrac{dy}{dx} = 3ax^2 + 2bx + c$, $\dfrac{d^2y}{dx^2} = 6ax + 2b$

Inflexion when $x = -1$: $\dfrac{d^2y}{dx^2} = 0 \Rightarrow 6a(-1) + 2b = 0 \Rightarrow -6a + 2b = 0$

Turning point when $x = 2$: $\dfrac{dy}{dx} = 0 \Rightarrow 3a \cdot 2^2 + 2b \cdot 2 + c = 0 \Rightarrow 12a + 4b + c = 0$

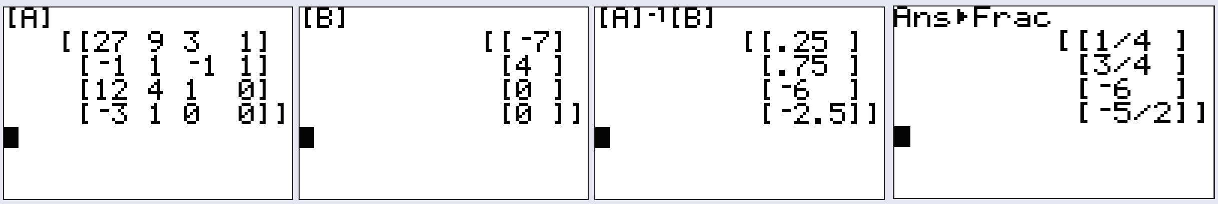
Passes through $(3, -7)$: $a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d = -7 \Rightarrow 27a + 9b + 3c + d = -7$

Passes through $(-1, 4)$: $a(-1)^3 + b(-1)^2 + c \cdot (-1) + d = 4 \Rightarrow -a + b - c + d = 4$

We have the following system of linear equations:

$$\begin{cases} 27a + 9b + 3c + d = -7 \\ -a + b - c + d = 4 \\ 12a + 4b + c = 0 \\ -3a + b = 0 \end{cases}$$

The system can be easily solved with a GDC:



So, $y = \dfrac{1}{4}x^3 + \dfrac{3}{4}x^2 - 6x - \dfrac{5}{2}$, and, at $x = 2$, $y = \dfrac{1}{4} \cdot 2^3 + \dfrac{3}{4} \cdot 2^2 - 6 \cdot 2 - \dfrac{5}{2} = -\dfrac{19}{2}$.

33 For the function $f(x) = 1 - \dfrac{9}{x^2} + \dfrac{18}{x^4} = 1 - 9x^{-2} + 18x^{-4}$:

$f'(x) = 18x^{-3} - 72x^{-5} = \dfrac{18(x^2 - 4)}{x^5} = \dfrac{18(x+2)(x-2)}{x^5} = 0 \Rightarrow x = -2, x = 2$

Since the curve is not defined for $x = 0$ (vertical asymptote), we have four intervals to check:

$f'(-3) = \dfrac{18\left((-3)^2 - 4\right)}{(-3)^5} = -\dfrac{10}{27} < 0 \Rightarrow f(x) \searrow$ on $(-\infty, -2)$

$f'(-1) = \dfrac{18\left((-1)^2 - 4\right)}{(-1)^5} = 54 > 0 \Rightarrow f(x) \nearrow$ on $(-2, 0)$

$f'(1) = \dfrac{18\left(1^2 - 4\right)}{1^5} = -54 < 0 \Rightarrow f(x) \searrow$ on $(0, 2)$

$f'(3) = \dfrac{18\left(3^2 - 4\right)}{3^5} = \dfrac{10}{27} > 0 \Rightarrow f(x) \nearrow$ on $(2, \infty)$

And because

$\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} 1 - \dfrac{9}{x^2} + \dfrac{18}{x^4} = 1 - 0 + 0 = 1$

$f(-2) = 1 - \dfrac{9}{(-2)^2} + \dfrac{18}{(-2)^4} = -\dfrac{1}{8}$

$f(2) = 1 - \dfrac{9}{2^2} + \dfrac{18}{2^4} = -\dfrac{1}{8}$

We conclude that both stationary points, $\left(-2, -\dfrac{1}{8}\right)$ and $\left(2, -\dfrac{1}{8}\right)$, are absolute minima.

34 a For the curve $y = \frac{1}{x} = x^{-1}$, $\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$, so the slope of the tangent at $(1, 1)$ is -1, and the equation of the tangent is: $y - 1 = -1 \cdot (x - 1) \Rightarrow y = -x + 2$.

b For the curve $y = \cos x$, $\frac{dy}{dx} = -\sin x$, so the slope of the tangent at $\left(\frac{\pi}{2}, 0\right)$ is $-\sin\left(\frac{\pi}{2}\right) = -1$, and the equation of the tangent is: $y - 0 = -1 \cdot \left(x - \frac{\pi}{2}\right) \Rightarrow y = -x + \frac{\pi}{2}$.

c Since $2 > \frac{\pi}{2} \Rightarrow -x + 2 > -x + \frac{\pi}{2}$, all the tangents on $y = \frac{1}{x}$ are **above** the tangents on $y = \cos x$, which means that $\frac{1}{x} > \cos x, 0 \leqslant x \leqslant \frac{\pi}{2}$.

35 For the curve $y = x^3 - x + 2$, the slope function is $\frac{dy}{dx} = 3x^2 - 1$.

If the point $(a, a^3 - a + 2)$ is a point of tangency, then the equation of the tangent is: $y - a^3 + a - 2 = (3a^2 - 1)(x - a)$.

The tangent should pass through the origin, so: $0 - a^3 + a - 2 = (3a^2 - 1)(0 - a) \Rightarrow -a^3 + a - 2 = -3a^3 + a \Rightarrow a^3 = 1$.

This gives us only one solution: $a = 1$.

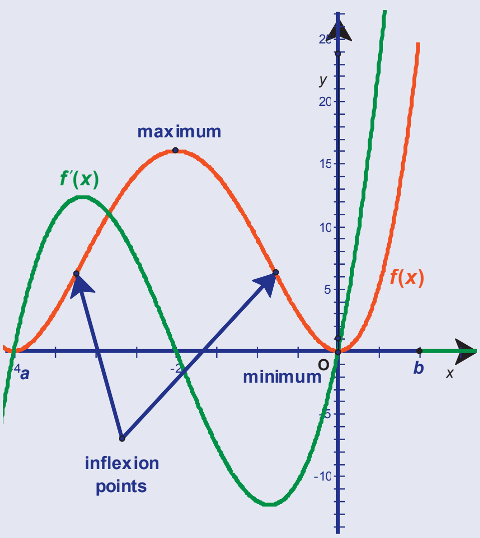
The tangent is: $y - 1^3 + 1 - 2 = (3 \cdot 1^2 - 1)(x - 1) \Rightarrow y = 2x$ at $(1, 2)$.

36 For the displacement function $s = s(t) = 50t - 10t^2 + 1000$:

a Velocity: $v(t) = \frac{ds}{dt} = 50 - 20t$

b $v(t) = 0 \Rightarrow 50 - 20t = 0 \Rightarrow t = \frac{5}{2}$, so the maximum displacement is: $s\left(\frac{5}{2}\right) = 50 \cdot \left(\frac{5}{2}\right) - 10 \cdot \left(\frac{5}{2}\right)^2 + 1000 = 1062.5$ m

37



Chapter 14

Practice questions

1 **a** $\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD} = -\overrightarrow{OC} + \overrightarrow{OD} = \overrightarrow{OD} - \overrightarrow{OC}$

b $\overrightarrow{OA} = \frac{1}{2}\overrightarrow{CD} = -\frac{1}{2}\left(-\overrightarrow{OC} + \overrightarrow{OD}\right) = \frac{1}{2}\left(\overrightarrow{OD} - \overrightarrow{OC}\right)$

c $\overrightarrow{AD} = -\overrightarrow{AO} + \overrightarrow{OD} = -\overrightarrow{OA} + \overrightarrow{OD} = -\frac{1}{2}\left(\overrightarrow{OD} - \overrightarrow{OC}\right) + \overrightarrow{OD} = \frac{1}{2}\overrightarrow{OD} + \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}\left(\overrightarrow{OD} + \overrightarrow{OC}\right)$

2 **a** $\mathbf{u} + 2\mathbf{v} = (-\mathbf{i} + 2\mathbf{j}) + 2(3\mathbf{i} + 5\mathbf{j}) = -\mathbf{i} + 2\mathbf{j} + 6\mathbf{i} + 10\mathbf{j} = 5\mathbf{i} + 12\mathbf{j}$

b The unit vector in the direction of $\mathbf{u} + 2\mathbf{v}$ is: $\frac{1}{\sqrt{5^2 + 12^2}}(5\mathbf{i} + 12\mathbf{j}) = \frac{1}{13}(5\mathbf{i} + 12\mathbf{j})$. So,

$\mathbf{w} = 26 \cdot \frac{1}{13}(5\mathbf{i} + 12\mathbf{j}) = 2(5\mathbf{i} + 12\mathbf{j}) = 10\mathbf{i} + 24\mathbf{j}$.

3 **a** $\left|\overrightarrow{OA}\right| = \sqrt{6^2 + 0^2} = 6$, so A lies on the circle.

$\left|\overrightarrow{OB}\right| = \sqrt{(-6)^2 + 0^2} = 6$, so B lies on the circle.

$\left|\overrightarrow{OC}\right| = \sqrt{5^2 + \sqrt{11}^2} = 6$, so C lies on the circle.

b $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ \sqrt{11} \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{11} \end{pmatrix}$

c Method I: Using a scalar product

$$\cos O\hat{A}C = \frac{\overrightarrow{AO} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AO}\right|\left|\overrightarrow{AC}\right|} = \frac{\begin{pmatrix} -6 \\ 0 \end{pmatrix}\begin{pmatrix} -1 \\ \sqrt{11} \end{pmatrix}}{6\sqrt{(-1)^2 + \sqrt{11}^2}} = \frac{6}{6\sqrt{12}} = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

Method II: Using a cosine rule in triangle OAC

In triangle OAC, SSS is given: $OA = OC = 6$, and $AC = \left|\begin{pmatrix} -1 \\ \sqrt{11} \end{pmatrix}\right| = \sqrt{12}$; hence,

$$\cos O\hat{A}C = \frac{6^2 + \left(\sqrt{12}\right)^2 - 6^2}{2 \cdot \sqrt{12}} = \frac{12}{2\sqrt{12}} = \frac{\sqrt{3}}{6}$$

d Method I: Using the result from **c**

Using the Pythagorean identity for sine, $\sin^2 \theta = 1 - \cos^2 \theta$, and the fact that sine is positive for angles from

$0 - 180^\circ$ we have: $\sin O\hat{A}C = \sqrt{1 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{1 - \frac{1}{12}} = \sqrt{\frac{11}{12}}$. Hence,

$A = \frac{1}{2}|AB|\,|AC| \sin \hat{A} = \frac{1}{2} 12 \cdot \sqrt{12}\sqrt{\frac{11}{12}} = 6\sqrt{11}$.

Method II: Finding the area using side and height dimensions

In triangle ABC, side $|AB| = 12$; the height on this side is the second coordinate of point C, so:

$A = \frac{1}{2} 12 \cdot \sqrt{11} = 6\sqrt{11}$.

Method III: Using the triangle in half circle property

The triangle ABC is a right triangle, with right angle in C. So: $A = \frac{1}{2}|AC|\,|AB| = \frac{1}{2}\sqrt{11} \cdot 12 = 6\sqrt{11}$.

4 **a** $\overrightarrow{OB} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2-5 \\ 7-1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

b The diagonals of quadrilateral $OABC$ are OB and AC. So, the angle between the diagonals is the same as the angle between vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$.

$\overrightarrow{OB} \cdot \overrightarrow{AB} = 10 \cdot (-3) + 5 \cdot (6) = 0$; hence, the diagonals are perpendicular, 90°.

5 $\mathbf{u} + \mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$

Then, $a(4\mathbf{i} + 3\mathbf{j}) = 8\mathbf{i} + (b-2)\mathbf{j} \Rightarrow \begin{cases} 4a = 8 \\ 3a = b-2 \end{cases} \Rightarrow \begin{cases} a = 2 \\ 6 + 2 = b \Rightarrow b = 8 \end{cases}$

6 The direction vector is $\begin{pmatrix} 3-(-1) \\ -1-4 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$; hence, the equation of the line is: $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 4 \\ -5 \end{pmatrix}$.

Note: For the direction vector we can use $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$, and for the initial point $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$. So, the equation of the line can be any of these combinations.

7 **a** The speed of the Toyundai: $\left|\begin{pmatrix} 18 \\ 24 \end{pmatrix}\right| = \sqrt{18^2 + 24^2} = 30$ km/h

The speed of the Chryssault: $\left|\begin{pmatrix} 36 \\ -16 \end{pmatrix}\right| = \sqrt{36^2 + (-16)^2} = \sqrt{1552} \approx 39.4$ km/h

b **i** After half an hour, the vehicles have covered half the distance: $\frac{1}{2}\begin{pmatrix} 18 \\ 24 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix}$; $\frac{1}{2}\begin{pmatrix} 36 \\ -16 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix}$

ii The vector joining their positions at 06:30 is $\begin{pmatrix} 9-18 \\ 12-(-8) \end{pmatrix} = \begin{pmatrix} -9 \\ 20 \end{pmatrix}$; hence, the distance between the vehicles is: $\left|\begin{pmatrix} -9 \\ 20 \end{pmatrix}\right| = \sqrt{9^2 + 20^2} = \sqrt{481} \approx 21.9$ km.

c The Toyundai must continue until its position vector is $\begin{pmatrix} 18 \\ k \end{pmatrix}$, so until $k = 24$. At that point, its position is $\begin{pmatrix} 18 \\ 24 \end{pmatrix}$. To reach this position, it must travel for a total of one hour. Hence, the crew start work at 07:00.

d The southern (Chryssault) crew lay: $800 \cdot 5 = 4000$ m of cable.

The northern (Toyundai) crew lay: $800 \cdot 4.5 = 3600$ m of cable.

Their starting points were $24 - (-8) = 32$ km apart; hence, they are now $32 - 3.6 - 4 = 24.4$ km apart.

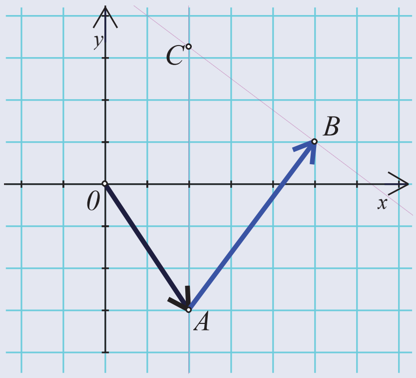
e The position vector of the Toyundai at 11:30 is $\begin{pmatrix} 18 \\ 24-3.6 \end{pmatrix} = \begin{pmatrix} 18 \\ 20.4 \end{pmatrix}$.

The distance to base camp is: $\left|\begin{pmatrix} 18 \\ 20.4 \end{pmatrix}\right| = \sqrt{18^2 + 20.4^2} = \sqrt{740.16} \approx 27.2$ km.

The time needed to cover this distance is: $\frac{27.2}{30} \cdot 60 = 54.4 \approx 54$ minutes.

8 The line passes through point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and has a direction vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Hence, its equation is: $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, or $\mathbf{r} = t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

9 **a**



b Point C has the same x-coordinate as A; hence, $C(2, y)$. We find the y-coordinate by using the fact that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular. From the graph, point B is (5, 1). So: $\overrightarrow{BC} = \begin{pmatrix} 2-5 \\ y-1 \end{pmatrix} = \begin{pmatrix} -3 \\ y-1 \end{pmatrix}$ and:

$0 = \overrightarrow{AB} \cdot \overrightarrow{BC} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} -3 \\ y-1 \end{pmatrix} = -9 + 4y - 4 = -13 + 4y \Rightarrow y = \frac{13}{4}$. Hence, $C\left(2, \frac{13}{4}\right)$. So, $\overrightarrow{OC} = \begin{pmatrix} 2 \\ \frac{13}{4} \end{pmatrix}$.

10 **a** **i** Initially, *Air One* is at position $\mathbf{r} = \begin{pmatrix} 16 \\ 12 \end{pmatrix}$; hence, its distance from the origin is: $\left|\begin{pmatrix} 16 \\ 12 \end{pmatrix}\right| = \sqrt{16^2 + 12^2} = 20$ km.

ii The velocity vector is $\mathbf{v} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$; hence, its speed is: $\left|\begin{pmatrix} 12 \\ -5 \end{pmatrix}\right| = \sqrt{12^2 + (-5)^2} = 13$ km/min.

b $\mathbf{r} = \begin{pmatrix} 16 \\ 12 \end{pmatrix} + t\begin{pmatrix} 12 \\ -5 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16+12t \\ 12-5t \end{pmatrix} \Rightarrow \begin{cases} x = 16+12t \\ y = 12-5t \end{cases}$

From the first equation, we have $t = \frac{x-16}{12}$. Substituting into the second equation:

$y = 12 - 5\frac{x-16}{12} = \frac{144 - 5x + 80}{12} \Rightarrow 12y = 224 - 5x \Rightarrow 5x + 12y = 224.$

Note: If we multiply the vector equation of the line by the vector perpendicular to the direction vector, we can find the result quite quickly.

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ 12 \end{pmatrix} + t\begin{pmatrix} 12 \\ -5 \end{pmatrix} \Rightarrow \begin{pmatrix} 5 \\ 12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}\begin{pmatrix} 16 \\ 12 \end{pmatrix} + t\begin{pmatrix} 5 \\ 12 \end{pmatrix}\begin{pmatrix} 12 \\ -5 \end{pmatrix}$. Now we have:

$5x + 12y = 5 \cdot 16 + 12 \cdot 12 + t \cdot 0 \Rightarrow 5x + 12y = 224.$

c We have to determine the angle between the direction vectors:

$\begin{pmatrix} 12 \\ -5 \end{pmatrix}\begin{pmatrix} 2.5 \\ 6 \end{pmatrix} = 12 \cdot 2.5 - 5 \cdot 6 = 0$; hence, the angle between the paths of the two aircraft is 90°.

d **i** $\mathbf{r} = \begin{pmatrix} 23 \\ -5 \end{pmatrix} + t\begin{pmatrix} 2.5 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 23+2.5t \\ -5+6t \end{pmatrix} \Rightarrow \begin{cases} x = 23 + 2.5t \Rightarrow t = \frac{x-23}{2.5} \\ y = -5 + 6t \Rightarrow t = \frac{y+5}{6} \end{cases}$

Hence, $\frac{x-23}{2.5} = \frac{y+5}{6}$.

Multiplying by 30: $30\frac{x-23}{2.5} = 30\frac{y+5}{6} \Rightarrow 12(x-23) = 5(y+5) \Rightarrow 12x - 5y = 301$

Note: We could also have used the method from **b**.

ii $\begin{cases} 5x + 12y = 224 \\ 12x - 5y = 301 \end{cases} \Rightarrow \begin{cases} 25x + 60y = 1120 \\ 144x - 60y = 3612 \end{cases} \Rightarrow 169x = 4732 \Rightarrow 28,\ y = \frac{12 \cdot 28 - 301}{5} = 7$

Hence, the paths cross at the point $(28, 7)$.

e We will determine the time at which each of the planes is at $(28, 7)$.

For *Air One*:

$$\begin{pmatrix} 28 \\ 7 \end{pmatrix} = \begin{pmatrix} 16+12t \\ 12-5t \end{pmatrix} \Rightarrow \begin{cases} 28-16=12t \\ 7-12=-5t \end{cases} \Rightarrow t=1$$

For *Air Two*:

$$\begin{pmatrix} 28 \\ 7 \end{pmatrix} = \begin{pmatrix} 23+2.5t \\ -5+6t \end{pmatrix} \Rightarrow \begin{cases} 28-23=2.5t \\ 7+5=6t \end{cases} \Rightarrow t=2$$

So, the planes are not at the point where the two paths cross at the same time, i.e. the planes do not collide.

11 $$\cos\theta = \frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 6 \\ -8 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right|\left|\begin{pmatrix} 6 \\ -8 \end{pmatrix}\right|} = \frac{6-16}{\sqrt{1+4}\sqrt{36+64}} = \frac{-10}{\sqrt{5}\cdot 10} = \frac{-1}{\sqrt{5}}$$

$$\theta = \cos^{-1}\frac{-1}{\sqrt{5}} = 116.565...° \approx 117°$$

12 Method I:

If (x, y) is a point on the line, then the vector $\begin{pmatrix} x-4 \\ y+1 \end{pmatrix}$ is the vector on the line, and it is perpendicular to the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Hence, their dot product is zero: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} x-4 \\ y+1 \end{pmatrix} = 0$

So, $0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} x-4 \\ y+1 \end{pmatrix} = 2(x-4)+3(y+1) = 2x-8+3y+3 = 2x+3y-5$ and the equation of the line is: $2x+3y=5$.

Method II:

If vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is perpendicular to the line, then the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, or $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, is a direction vector of the line.

So, a vector equation of the line is $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Now, we have to transform the equation into Cartesian form: $\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4+3t \\ -1-2t \end{pmatrix} \Rightarrow \begin{cases} x = 4+3t \quad /\cdot 2 \\ y=-1-2t \quad /\cdot 3 \end{cases} \Rightarrow \begin{cases} 2x = 8+6t \\ 3y=-3-6t \end{cases} \Rightarrow 2x+3y=5$

13 **a** At 13:00, $t=1$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + 1\cdot\begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 6 \\ 20 \end{pmatrix}$

b **i** The velocity vector is: $\begin{pmatrix} x \\ y \end{pmatrix}_{t=1} - \begin{pmatrix} x \\ y \end{pmatrix}_{t=0} = \begin{pmatrix} 6 \\ 20 \end{pmatrix} - \begin{pmatrix} 0 \\ 28 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}$

ii The speed is the magnitude of the velocity vector; therefore: $\left|\begin{pmatrix} 6 \\ -8 \end{pmatrix}\right| = \sqrt{6^2+(-8)^2} = 10$ km/h

c $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + t\cdot\begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 6t \\ 28-8t \end{pmatrix} \Rightarrow \begin{cases} x=6t \quad /\cdot 4 \\ y=28-8t \quad /\cdot 3 \end{cases} \Rightarrow \begin{cases} 4x=24t \\ 3y=84-24t \end{cases} \Rightarrow 4x+3y=84$

d The two ships will collide if the point (18, 4) is on the line. So:

$$\begin{pmatrix} 18 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + t\cdot\begin{pmatrix} 6 \\ -8 \end{pmatrix} \Rightarrow \begin{cases} 18=6t \\ 4=28-8t \end{cases} \Rightarrow t=3$$

Therefore, the ships will collide at $t=12+3=15{:}00$ hours.

e $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 4 \end{pmatrix} + (t-1)\cdot\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 18+5t-5 \\ 4+12t-12 \end{pmatrix} = \begin{pmatrix} 13+5t \\ -8+12t \end{pmatrix} = \begin{pmatrix} 13 \\ -8 \end{pmatrix} + t\begin{pmatrix} 5 \\ 12 \end{pmatrix}$

f At $t = 3$, *Aristides* is at $\begin{pmatrix} 18 \\ 4 \end{pmatrix}$ and *Boadicea* is at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ -8 \end{pmatrix} + 3 \cdot \begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 28 \\ 28 \end{pmatrix}$.

Therefore, their distance vector is: $\begin{pmatrix} 28 \\ 28 \end{pmatrix} - \begin{pmatrix} 18 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 24 \end{pmatrix}$; hence, the ships are $\sqrt{10^2 + 24^2} = \sqrt{676} = 26$ km apart.

14 $\cos\theta = \dfrac{\mathbf{ab}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{(-4\mathbf{i} - 2\mathbf{j})(\mathbf{i} - 7\mathbf{j})}{|-4\mathbf{i} - 2\mathbf{j}|\,|\mathbf{i} - 7\mathbf{j}|} = \dfrac{-4 + 14}{\sqrt{16+4}\sqrt{1+49}} = \dfrac{10}{\sqrt{20}\sqrt{50}} = \dfrac{10}{10\sqrt{2}\sqrt{5}} = \dfrac{1}{\sqrt{10}}$

$\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{10}}\right) = 71.565\,05...^\circ = 72^\circ$

15 a At $t = 2$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0.7 \\ 1 \end{pmatrix} = \begin{pmatrix} 3.4 \\ 2 \end{pmatrix}$

The distance from the point $(0, 0)$ is: $\sqrt{3.4^2 + 2^2} = \sqrt{15.56} \approx 3.94$ m

b The speed of the car is: $\left\|\begin{pmatrix} 0.7 \\ 1 \end{pmatrix}\right\| = \sqrt{0.7^2 + 1} = \sqrt{1.49} \approx 1.22$ m/s

c $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0.7 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 + 0.7t \\ t \end{pmatrix} \Rightarrow t = y$; hence: $x = 2 + 0.7y \Rightarrow x - 0.7y = 2$

d Solve the system of equations:

$\begin{cases} x - 0.7y = 2 \\ y = 0.6x + 2 \end{cases} \Rightarrow x - 0.42x - 1.4 = 2 \Rightarrow x = \dfrac{170}{29} \approx 5.86$

$y = \dfrac{160}{29} = 5.52$

So, the collision point is $\left(\dfrac{170}{29}, \dfrac{160}{29}\right)$.

e Since $y = t$, the time of collision is 5.52 seconds.

The distance covered by the motorcycle is: $\left\|\begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 5.86 \\ 5.52 \end{pmatrix}\right\| = \sqrt{5.86^2 + 3.52^2} \approx 6.84$ m. Therefore, the speed is the quotient of distance and time, and we have: $\dfrac{6.84}{5.52} \approx 1.24$ m/s.

16 The direction vector is $\begin{pmatrix} 6-1 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$; hence, the equation of the line is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

Note: We can use the other point as the initial point and then we will have the equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix} + t\begin{pmatrix} -5 \\ -2 \end{pmatrix}$.

17 a $\begin{pmatrix} 2x \\ x-5 \end{pmatrix}\begin{pmatrix} x-1 \\ 5 \end{pmatrix} = 0 \Rightarrow 2x^2 + 2x + 5x - 25 = 0 \Rightarrow 2x^2 + 7x - 25 = 0$

b $2x^2 + 7x - 25 = 0 \Rightarrow x_1 = \dfrac{-7 - \sqrt{249}}{4} \approx -5.69,\ x_2 = \dfrac{-7 - \sqrt{249}}{4} \approx 2.19$

IB exam question:

a $\begin{pmatrix} 2x \\ x-3 \end{pmatrix}\begin{pmatrix} x+1 \\ 5 \end{pmatrix} = 0 \Rightarrow 2x^2 + 2x + 5x - 15 = 0 \Rightarrow 2x^2 + 7x - 15 = 0$

b $2x^2 + 7x - 15 = 0 \Rightarrow x_1 = -5,\ x_2 = \dfrac{3}{2}$

18 a i $\overrightarrow{OA} = \begin{pmatrix} 240 \\ 70 \end{pmatrix} \Rightarrow \left|\overrightarrow{OA}\right| = \sqrt{240^2 + 70^2} = 250$. So, the unit vector is: $\dfrac{1}{250}\begin{pmatrix} 240 \\ 70 \end{pmatrix} = \begin{pmatrix} \frac{24}{25} \\ \frac{7}{25} \end{pmatrix} = \begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix}$.

ii $\mathbf{v} = 300\begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix} = \begin{pmatrix} 288 \\ 84 \end{pmatrix}$

iii $t = \frac{250}{300} = \frac{5}{6}$ hour, or 50 minutes

b $\overrightarrow{AB} = \begin{pmatrix} 480-240 \\ 250-70 \end{pmatrix} = \begin{pmatrix} 240 \\ 180 \end{pmatrix}$

$$\cos\theta = \frac{\begin{pmatrix} 240 \\ 70 \end{pmatrix}\begin{pmatrix} 240 \\ 180 \end{pmatrix}}{\left|\begin{pmatrix} 240 \\ 70 \end{pmatrix}\right|\left|\begin{pmatrix} 240 \\ 180 \end{pmatrix}\right|} = \frac{57\,600 + 12\,600}{\sqrt{240^2+70^2}\sqrt{240^2+180^2}} = \frac{70\,200}{250 \cdot 300} = 0.936$$

So, $\theta = \cos^{-1} 0.936 = 20.609...^\circ \approx 20.6^\circ$.

c **i** $\overrightarrow{AX} = \begin{pmatrix} 339-240 \\ 238-70 \end{pmatrix} = \begin{pmatrix} 99 \\ 168 \end{pmatrix}$

ii $\begin{pmatrix} -3 \\ 4 \end{pmatrix}\begin{pmatrix} 240 \\ 180 \end{pmatrix} = -3 \cdot 240 + 4 \cdot 180 = 0$; hence, $\mathbf{n} \perp \overrightarrow{AB}$.

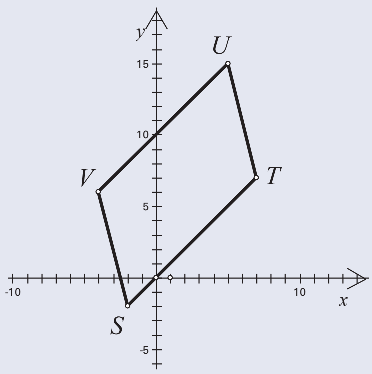
iii The scalar projection of $\overrightarrow{AX}$ in the direction of $\mathbf{n}$ is $\frac{1}{5}\begin{pmatrix} 99 \\ 168 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \frac{-297+672}{5} = 75$; hence, the distance XY is 75 km.

d Using Pythagoras' theorem, we can find the distance from A to Y using the distances AX and XY So, $AX = \sqrt{99^2+168^2} = \sqrt{38\,025} = 195$; hence, $AY = \sqrt{195^2 - 75^2} = \sqrt{32\,400} = 180$ km.

19 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix} \Rightarrow \begin{cases} x = 1-2t \Rightarrow \frac{x-1}{-2} = t \\ y = 2+3t \Rightarrow \frac{y-2}{3} = t \end{cases} \Rightarrow \frac{x-1}{-2} = \frac{y-2}{3}$

$3(x-1) = -2(y-2) \Rightarrow 3x+2y = 4+3 \Rightarrow 3x+2y = 7$

20



a $\overrightarrow{ST} = \begin{pmatrix} 7-(-2) \\ 7-(-2) \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$, and, since $STUV$ is a parallelogram, $\overrightarrow{VU} = \overrightarrow{ST} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$.

$\overrightarrow{VU} = \mathbf{u} - \mathbf{v} = \begin{pmatrix} 9 \\ 9 \end{pmatrix} \Rightarrow \begin{pmatrix} 5 \\ 15 \end{pmatrix} - \mathbf{v} = \begin{pmatrix} 9 \\ 9 \end{pmatrix} \Rightarrow \begin{pmatrix} 5 \\ 15 \end{pmatrix} - \begin{pmatrix} 9 \\ 9 \end{pmatrix} = \mathbf{v} \Rightarrow \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \mathbf{v}$, and the coordinates of V are: $(-4, 6)$.

b The line contains the point (5, 15) and the direction vector is parallel to $\begin{pmatrix} 9 \\ 9 \end{pmatrix}$. So, for the direction vector, we can use the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So, $\mathbf{r} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Note: We can also use the direction vector $\begin{pmatrix} 9 \\ 9 \end{pmatrix}$ and initial point $(-4, 6)$. However, the question directs us to use point U. This also ensures we avoid any possible mistakes we made when finding the coordinates of V.

c $\begin{pmatrix} 1 \\ 11 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} 1 = 5 + \lambda \Rightarrow \lambda = -4 \\ 11 = 15 + \lambda \Rightarrow \lambda = -4 \end{cases}$

So, the point is on the line when $\lambda = -4$.

d i $\overrightarrow{EW} = \begin{pmatrix} a-1 \\ 17-11 \end{pmatrix} = \begin{pmatrix} a-1 \\ 6 \end{pmatrix}$

$\left|\overrightarrow{EW}\right| = \sqrt{(a-1)^2 + 36} = \sqrt{a^2 - 2a + 37} = 2\sqrt{13} \Rightarrow a^2 - 3a + 37 = 52$

So: $a^2 - 2a - 15 = 0 \Rightarrow a_1 = -3, a_2 = 5$

ii For $a = -3$: $\overrightarrow{EW} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \overrightarrow{ET} = \begin{pmatrix} 7-1 \\ 1-11 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$

So, $\cos\theta = \dfrac{\begin{pmatrix} -4 \\ 6 \end{pmatrix}\begin{pmatrix} 6 \\ -4 \end{pmatrix}}{\left|\begin{pmatrix} -4 \\ 6 \end{pmatrix}\right|\left|\begin{pmatrix} 6 \\ -4 \end{pmatrix}\right|} = \dfrac{-24-24}{\sqrt{16+36}\sqrt{16+36}} = \dfrac{-48}{52} = -\dfrac{12}{13}$, and $\theta = \cos^{-1}\left(-\dfrac{12}{13}\right) = 157.38...^\circ \approx 157^\circ$.

21 The angle between the lines is the angle between their direction vectors. The direction vector of the first line is $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, and the second is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The angle is: $\cos\theta = \dfrac{\begin{pmatrix} 4 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\left|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right|\left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right|} = \dfrac{4-3}{\sqrt{16+9}\sqrt{1+1}} = \dfrac{1}{5\sqrt{2}}$. So,

$\theta = \cos^{-1}\left(\dfrac{1}{5\sqrt{2}}\right) = 81.869...^\circ \approx 81.9^\circ$.

Note: If the cosine of the angle was negative, then the angle would be obtuse; so, to find the acute angle, we would have to subtract the angle from 180°. Here, it was not the case.

22 a $|\mathbf{a}| = \sqrt{12^2 + 5^2} = 13$

b $|\mathbf{b}| = \sqrt{6^2 + 8^2} = 10$

The unit vector in the direction of **b** is: $\dfrac{1}{10}(6\mathbf{i} + 8\mathbf{j}) = 0.6\mathbf{i} + 0.8\mathbf{j}$.

c $\cos\theta = \dfrac{\mathbf{ab}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{12\cdot 6 + 5\cdot 8}{13\cdot 10} = \dfrac{112}{13\cdot 10} = \dfrac{56}{65}$

23 The coordinates of the point of intersection should satisfy both equations. So:

$\begin{pmatrix} 5+3\lambda \\ 1-2\lambda \end{pmatrix} = \begin{pmatrix} -2+4t \\ 2+t \end{pmatrix} \Rightarrow \begin{cases} 3\lambda - 4t = -7 \\ -2\lambda - t = 1 \Rightarrow t = -2\lambda - 1 \end{cases} \Rightarrow 3\lambda + 8\lambda + 4 = -7 \Rightarrow \lambda = -1, t = 1$

Therefore, the position vector of the point is: $\overrightarrow{OP} = \begin{pmatrix} 5-3 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Note: We can transform the vector equations to Cartesian form ($2x + 3y = 13$, and $x - 4y = -10$) and then solve the system.

24 a $\overrightarrow{OR} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 10 \\ 1 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

b $\cos O\hat{P}Q = \dfrac{\overrightarrow{PO}\,\overrightarrow{PQ}}{\left|\overrightarrow{PO}\right|\left|\overrightarrow{PQ}\right|} = \dfrac{\begin{pmatrix} -7 \\ -3 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}}{\sqrt{49+9}\sqrt{9+4}} = \dfrac{-21+6}{\sqrt{58}\sqrt{13}} = \dfrac{-15}{\sqrt{754}}$

c **i** Since $P\hat{Q}R + O\hat{P}Q = 180^\circ$, $\cos P\hat{Q}R = \cos\left(180^\circ - O\hat{P}Q\right) = -\cos O\hat{P}Q$.

ii Using the Pythagorean identity for sine and the fact that the sine of angles in a triangle is always positive, we have:

$$\sin P\hat{Q}R = \sqrt{1-\cos^2 P\hat{Q}R} = \sqrt{1-\cos^2 O\hat{P}Q} = \sqrt{1-\frac{15^2}{754}} = \sqrt{\frac{529}{754}} = \frac{23}{\sqrt{754}}$$

iii Area of the parallelogram: $A = \left|\overrightarrow{OR}\right|\left|\overrightarrow{OP}\right|\sin\theta = \left|\begin{pmatrix}3\\-2\end{pmatrix}\right|\left|\begin{pmatrix}7\\3\end{pmatrix}\right| = \sqrt{13}\sqrt{53}\,\frac{23}{\sqrt{13\cdot 53}} = 23$ units^2

25 **a** $\overrightarrow{OB} = \begin{pmatrix}-1\\7\end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix}8\\9\end{pmatrix}$

b To find D, we have to find the vector of the side of the parallelogram:

$\overrightarrow{AD} = \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}8\\9\end{pmatrix} - \begin{pmatrix}-1\\7\end{pmatrix} = \begin{pmatrix}9\\2\end{pmatrix}$. Now we can find the position vector of D:

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix}2\\2\end{pmatrix} + \begin{pmatrix}9\\2\end{pmatrix} = \begin{pmatrix}11\\4\end{pmatrix}$. Hence, $d = 11$.

c $\overrightarrow{BD} = \begin{pmatrix}11\\4\end{pmatrix} - \begin{pmatrix}-1\\7\end{pmatrix} = \begin{pmatrix}12\\-3\end{pmatrix}$

d **i** $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\7\end{pmatrix} + t\,\overrightarrow{BD} = \begin{pmatrix}-1\\7\end{pmatrix} + t\begin{pmatrix}12\\-3\end{pmatrix}$

Note: For the direction vector, we can use $\begin{pmatrix}4\\-1\end{pmatrix}$. Then the equation would be $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\7\end{pmatrix} + t\begin{pmatrix}4\\-1\end{pmatrix}$.

ii At point B, $t = 0$. We can see that $\begin{pmatrix}-1\\7\end{pmatrix} = \begin{pmatrix}-1\\7\end{pmatrix} + 0\cdot\begin{pmatrix}12\\-3\end{pmatrix}$.

e $$\begin{pmatrix}7\\5\end{pmatrix} = \begin{pmatrix}-1\\7\end{pmatrix} + t\begin{pmatrix}12\\-3\end{pmatrix} \Rightarrow \left\{\begin{array}{l} 7 = -1 + 12t \Rightarrow t = \dfrac{8}{12} \\ 5 = 7 - 3t \Rightarrow t = \dfrac{-2}{-3} \end{array}\right. \Rightarrow t = \frac{2}{3}$$

f $\overrightarrow{CP} = \overrightarrow{OP} - \overrightarrow{OC} = \begin{pmatrix}7\\5\end{pmatrix} - \begin{pmatrix}8\\9\end{pmatrix} = \begin{pmatrix}-1\\-4\end{pmatrix}$

$\overrightarrow{CP}\cdot\overrightarrow{BD} = \begin{pmatrix}-1\\-4\end{pmatrix}\begin{pmatrix}12\\-3\end{pmatrix} = -12 + 12 = 0$; hence, $\overrightarrow{CP} \perp \overrightarrow{BD}$.

26 **a** **i** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}-3\\-1\end{pmatrix} - \begin{pmatrix}2\\-2\end{pmatrix} = \begin{pmatrix}-5\\1\end{pmatrix}$

ii $\left|\overrightarrow{AB}\right| = \sqrt{25+1} = \sqrt{26}$

b $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix}d\\23\end{pmatrix} - \begin{pmatrix}2\\-2\end{pmatrix} = \begin{pmatrix}d-2\\25\end{pmatrix}$

c **i** $B\hat{A}D = 90^\circ$; hence, $\overrightarrow{AB} \perp \overrightarrow{AD} \Rightarrow \overrightarrow{AB}\cdot\overrightarrow{AD} = 0$.

$0 = \overrightarrow{AB}\cdot\overrightarrow{AD} = \begin{pmatrix}-5\\1\end{pmatrix}\begin{pmatrix}d-2\\25\end{pmatrix} = -5d + 10 + 25 \Rightarrow d = 7$

ii $\overrightarrow{OD} = \begin{pmatrix}d\\23\end{pmatrix} = \begin{pmatrix}7\\23\end{pmatrix}$

d $\overrightarrow{BC} = \overrightarrow{AD} = \begin{pmatrix} 7-2 \\ 25 \end{pmatrix} = \begin{pmatrix} 5 \\ 25 \end{pmatrix}\begin{pmatrix} 5 \\ 25 \end{pmatrix}$

$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} + \begin{pmatrix} 5 \\ 25 \end{pmatrix} = \begin{pmatrix} 2 \\ 24 \end{pmatrix}$

e The area is the product of the sides: $A = \left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{AD}\right| = \sqrt{25+1}\sqrt{25+625} = \sqrt{26}\sqrt{650} = 130$ units^2.

27 a i $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (-5\mathbf{i} - 5\mathbf{j}) - (\mathbf{i} - 3\mathbf{j}) = -6\mathbf{i} - 2\mathbf{j}$

ii $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \overrightarrow{OA} + \overrightarrow{BC} = (4\mathbf{i} + 2\mathbf{j}) + (-6\mathbf{i} - 2\mathbf{j}) = -2\mathbf{i}$

b $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = -2\mathbf{i} - (\mathbf{i} - 3\mathbf{j}) = -3\mathbf{i} + 3\mathbf{j}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-5\mathbf{i} - 5\mathbf{j}) - (4\mathbf{i} + 2\mathbf{j}) = -9\mathbf{i} - 7\mathbf{j}$

$\cos\theta = \dfrac{\overrightarrow{BD} \cdot \overrightarrow{AC}}{\left|\overrightarrow{BD}\right|\left|\overrightarrow{AC}\right|} = \dfrac{27-21}{\sqrt{9+9}\sqrt{81+49}} = \dfrac{6}{\sqrt{18}\sqrt{130}} = \dfrac{6}{\sqrt{2340}} \Rightarrow \theta = 82.8749...^\circ \approx 82.9^\circ$

c $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + t(2\mathbf{i} + 7\mathbf{j})$

d We have to solve the vector equation: $\mathbf{i} - 3\mathbf{j} + t(2\mathbf{i} + 7\mathbf{j}) = 4\mathbf{i} + 2\mathbf{j} + s(\mathbf{i} + 4\mathbf{j})$.

Hence, $\begin{cases} 1 + 2t = 4 + s \\ -3 + 7t = 2 + 4s \end{cases} \Rightarrow s = -3 + 2t \Rightarrow -3 + 7t = 2 - 12 + 8t \Rightarrow t = 7, s = 11$. So, the position vector of the intersection is: $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + 7 \cdot (2\mathbf{i} + 7\mathbf{j}) = 15\mathbf{i} + 46\mathbf{j}$.

28 a $\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BG} = 5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$

b $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = -5\mathbf{i} + 5\mathbf{k}$

c $\overrightarrow{EB} = \overrightarrow{EF} + \overrightarrow{FG} + \overrightarrow{GB} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$

29 a The unit vector in the direction of $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ is: $\dfrac{1}{\sqrt{3^2 + 4^2 + 0^2}}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$, so the velocity vector of the balloon is: $\dfrac{18}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$. Therefore, the equation of the path of the balloon is the same as the equation of the line through $(0, 0, 5)$ with direction vector $\dfrac{18}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$: $\mathbf{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + t \cdot \dfrac{18}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$.

b i $t = 0 \Rightarrow (49, 32, 0)$

ii The velocity vector is $\begin{pmatrix} -48 \\ -24 \\ 6 \end{pmatrix}$, so the speed is: $\left|\begin{pmatrix} -48 \\ -24 \\ 6 \end{pmatrix}\right| = \sqrt{48^2 + 24^2 + 6^2} = \sqrt{2916} = 54$ km/h.

c i At R: $\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + t \cdot \dfrac{18}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 49 \\ 32 \\ 0 \end{pmatrix} + t\begin{pmatrix} -48 \\ -24 \\ 6 \end{pmatrix} \Rightarrow \begin{cases} t\dfrac{18 \cdot 3}{5} = 49 - 48t \\ t\dfrac{18 \cdot 4}{5} = 32 - 24t \\ 5 = 6t \end{cases}$

$t = \dfrac{5}{6}$ hour (50 minutes) satisfies all three equations.

ii Substituting $t = \dfrac{5}{6}$ into the expression for h (or b): $\begin{pmatrix} 49 \\ 32 \\ 0 \end{pmatrix} + \dfrac{5}{6}\begin{pmatrix} -48 \\ -24 \\ 6 \end{pmatrix} = \begin{pmatrix} 49 - 40 \\ 32 - 20 \\ 0 + 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \\ 5 \end{pmatrix}$.

Hence, $R(9, 12, 5)$.

30 a i $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 200 \\ 400 \end{pmatrix} - \begin{pmatrix} -600 \\ -200 \end{pmatrix} = \begin{pmatrix} 800 \\ 600 \end{pmatrix}$

ii $|\overrightarrow{AB}| = \sqrt{800^2 + 600^2} = 1000$; hence, the unit vector is: $\frac{1}{1000}\begin{pmatrix} 800 \\ 600 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$.

b i $\mathbf{v} = 250\begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 200 \\ 150 \end{pmatrix}$

ii At 13:00, $t = 1$, so: $\begin{pmatrix} -600 \\ -200 \end{pmatrix} + 1 \cdot \begin{pmatrix} 200 \\ 150 \end{pmatrix} = \begin{pmatrix} -400 \\ -50 \end{pmatrix}$

iii The distance from A to B is 1000 km, and, since the velocity of the aircraft is 250 km/h, the time is $\frac{1000}{250} = 4$ hours; hence, the aircraft is flying over town B at 16:00.

c Method I: Evaluating the time needed

Time taken to travel from A to B to C is 9 hours $\left(\frac{81}{9}\text{hours}\right)$. The warning light will go on after 16 000 litres of fuel have been used. Time taken to use 16 000 litres $= \frac{16\,000}{1800} = \frac{80}{9}$. Hence, $\frac{1}{9}$ hour remains and the distance to town C is: $\frac{1}{9}250 \approx 27.8$ km.

Method II: Evaluating the distances needed

The distance from A to B to C is 2250 km. The distance covered using 16 000 litres of fuel is:

$\frac{16\,000}{1800}250 \approx 2222.22$ km. So, the distance to town C is $2250 - 2222.22 \approx 27.8$ km.

Method III: Evaluating fuel usage

Fuel used from A to B $= 1800 \cdot 4 = 7200$ litres.

Fuel remaining until the light goes on $= 16\,000 - 7200 = 8800$ litres.

Number of hours before the warning light goes on: $\frac{8800}{1800} = 4\frac{8}{9}$ hours; therefore, the time remaining is $\frac{1}{9}$ hour, and the distance to town C is: $\frac{1}{9}250 \approx 27.8$ km.

31 a The vectors are perpendicular if their scalar product is zero. So, firstly, we have to find the vectors:

$$\overrightarrow{QR} = \begin{pmatrix} 1-3 \\ 0-3 \\ 2c-5 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 2c-5 \end{pmatrix}, \quad \overrightarrow{PR} = \begin{pmatrix} 1-4 \\ 0-1 \\ 2c+1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 2c+1 \end{pmatrix}$$

$$\overrightarrow{QR} \cdot \overrightarrow{PR} = \begin{pmatrix} -2 \\ -3 \\ 2c-5 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \\ 2c+1 \end{pmatrix} = 6 + 3 + (2c-5)(2c+1) = 4c^2 - 8c + 4$$

The vectors are perpendicular if: $4c^2 - 8c + 4 = 0 \Rightarrow 4(c-1)^2 = 0 \Rightarrow c = 1$

b $$\overrightarrow{PR} = \begin{pmatrix} -3 \\ -1 \\ 2(1)+1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix}, \quad \overrightarrow{PS} = \begin{pmatrix} 1-4 \\ 1-1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$$

$$\overrightarrow{PS} \times \overrightarrow{PR} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

c A vector equation of the line is:

$$\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3-3t \\ 3-t \\ 5+3t \end{pmatrix}, \; t \in \mathbb{R}$$

d We need one more direction vector (which is not parallel to the direction vector of the line) to determine a normal to the plane. We will take a point on the line and point S: $\overrightarrow{SQ} = \begin{pmatrix} 3-1 \\ 3-1 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$. Hence, the normal will be:

$$\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 3 \\ -3 & -1 & 3 \end{vmatrix} = \begin{pmatrix} 9 \\ -15 \\ 4 \end{pmatrix}.$$

Therefore, the equation will be: $9(x-1) - 15(y-1) + 4(z-2) = 0 \Rightarrow 9x - 15y + 4z = 2$.

Note: We have a point and two vectors in the plane, so we can write parametric equations of the plane:

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \lambda, \mu \in \mathbb{R}$$

e Method I:

Shortest distance is: $\dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|}$

Since $\overrightarrow{PQ} = \begin{pmatrix} 3-4 \\ 3-1 \\ 5+1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}$, we have: $\dfrac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|} = \dfrac{\left|\begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}\begin{pmatrix} 9 \\ -15 \\ 4 \end{pmatrix}\right|}{\sqrt{81+225+16}} = \dfrac{15}{\sqrt{322}}$

Method II:

We can use the distance formula for a point (x_0, y_0, z_0) and a plane $ax + by + cz + d = 0$:

$$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Hence, for the point $P(4, 1, -1)$ and the plane $9x - 15y + 4z - 2 = 0$, the distance is:

$$d = \frac{|9(4) - 15(1) + 4(-1) - 2|}{\sqrt{9^2 + 15^2 + 4^2}} = \frac{15}{\sqrt{322}}.$$

32 a $\overrightarrow{AB} = \begin{pmatrix} 0-1 \\ -1-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 1-0 \\ 0+1 \\ 2-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

b $\overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -3 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

c We can use the formula for the area of a triangle: $A = \frac{1}{2}|\mathbf{a} \cdot \mathbf{b}|$. Hence: $A = \frac{1}{2}\left|\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right| = \frac{1}{2}\sqrt{1+1+4} = \frac{\sqrt{6}}{2}$.

d A normal to the plane is $\vec{n} = \overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. Since point A is on the plane, the equation is:

$-1(x-1) + 1(y-2) + 2(z-1) = 0 \Rightarrow -x + y + 2z = 3$

e The normal **n** is parallel to the required line. Hence,

$x = 2 - t$

$y = -1 + t$, where $t \in \mathbb{R}$.

$z = -6 + 2t$

f The distance formula for a point (x_0, y_0, z_0) and a plane $ax+by+cz+d=0$ is: $d=\dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.

Hence, for the point $(2,-1-6)$ and the plane $-x+y+2z-3=0$, the distance is:

$$d=\frac{|(-1)2+1(-1)+2(-6)-3|}{\sqrt{1+1+4}}=\frac{18}{\sqrt{6}}=3\sqrt{6}$$

g Since $|\mathbf{n}|=\sqrt{1+1+4}=\sqrt{6}$, a unit vector in the direction of $\mathbf{n}$ is: $\dfrac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}$.

h Firstly, we will find the point of intersection of the plane and the line through D perpendicular to the plane. Hence, we have to find the intersection of the plane P and the line from part **e**:

Since $x=2-t, y=-1+t, z=-6+2t$, we have:

$-(2-t)+(-1+t)+2(-6+2t)=3\Rightarrow 6t=18\Rightarrow t=3$. So, the point of intersection is $(-1,2,0)$. This point is the midpoint between points D and E. Hence: $(-1,2,0)=\left(\dfrac{x_E+2}{2},\dfrac{y_E-1}{2},\dfrac{z_E-6}{2}\right)\Rightarrow E(-4,5,6)$.

33 a $\mathbf{u}\times\mathbf{v}=\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\2&-1&2\end{vmatrix}=\mathbf{i}\begin{vmatrix}2&3\\-1&2\end{vmatrix}-\mathbf{j}\begin{vmatrix}1&3\\2&2\end{vmatrix}+k\begin{vmatrix}1&2\\2&-1\end{vmatrix}=\begin{pmatrix}7\\4\\-5\end{pmatrix}$

b Method I:

$$\mathbf{w}=\begin{pmatrix}\lambda+2\mu\\2\lambda-\mu\\3\lambda+2\mu\end{pmatrix}$$

The line of intersection of the planes is parallel to $\mathbf{u}\times\mathbf{v}$. So,

$$\mathbf{w}(\mathbf{u}\times\mathbf{v})=\begin{pmatrix}\lambda+2\mu\\2\lambda-\mu\\3\lambda+2\mu\end{pmatrix}\begin{pmatrix}7\\4\\-5\end{pmatrix}=7\lambda+14\mu+8\lambda-4\mu-15\lambda-10\mu=0 \text{ (for all } \lambda,\mu\text{)}.$$

Hence, **w** is perpendicular to the line of intersection.

Method II:

The line of intersection is perpendicular to the normals of both planes; hence, on vectors **u** and **v**. Therefore, it will be perpendicular to the plane containing those two vectors, that is, to all vectors of the form $\lambda\mathbf{u}+\mu\mathbf{v}=\mathbf{w}$.

Method III:

The line of intersection is perpendicular to the normals of both planes; hence, on vectors **u** and **v**. Therefore, for a direction vector **d** of the line, it holds:

$$\begin{cases}\mathbf{d}\cdot\mathbf{u}=0\\\mathbf{d}\cdot\mathbf{v}=0\end{cases}\Rightarrow\mathbf{d}(\lambda\mathbf{u}+\mu\mathbf{v})=\lambda\mathbf{d}\cdot\mathbf{u}+\mu\mathbf{d}\cdot\mathbf{v}=0;\text{ and }\mathbf{d}\text{ is perpendicular to }\mathbf{w}.$$

34 a $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{OB}=\begin{pmatrix}2+2\\1-1\\-2-1\end{pmatrix}=\begin{pmatrix}4\\0\\-3\end{pmatrix}\Rightarrow P(4,0,-3)$

$\overrightarrow{OQ}=\overrightarrow{OA}+\overrightarrow{OC}=\begin{pmatrix}2+1\\1+2\\-2+2\end{pmatrix}=\begin{pmatrix}3\\3\\0\end{pmatrix}\Rightarrow Q(3,3,0)$

$\overrightarrow{OR}=\overrightarrow{OB}+\overrightarrow{OC}=\begin{pmatrix}2+1\\-1+2\\-1+2\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}\Rightarrow R(3,1,1)$

$$\overrightarrow{OS} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \begin{pmatrix} 2+2+1 \\ 1-1+2 \\ -2-1+2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \Rightarrow S(5, 2, -1)$$

b $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} = -\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$; hence, the equation of the plane is:

$$3(x-2) + 2(y-1) + 4(z+2) = 0 \Rightarrow 3x + 2y + 4z = 0$$

Note: Parametric equations of the plane are:

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \lambda, \mu \in \mathbb{R}$$

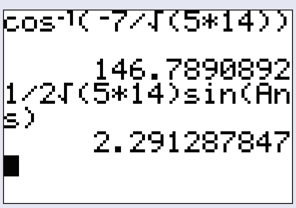
c $V = \left|\left(\overrightarrow{OA} \times \overrightarrow{OB}\right) \cdot \overrightarrow{OC}\right| = \left|\begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right| = |-3-4-8| = 15$

35 a $\overrightarrow{AB} = \begin{pmatrix} -1+1 \\ 3-2 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 0+1 \\ -1-2 \\ 1-3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$

$$\cos\theta = \frac{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}}{\sqrt{1+4}\sqrt{1+9+4}} = \frac{-7}{\sqrt{5}\sqrt{14}} \Rightarrow \theta = 146.789...^\circ \approx 147^\circ$$

b Method I:

$$A = \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| \sin\theta = \frac{1}{2}\sqrt{5}\sqrt{14}\,\sin 146.789...^\circ = 2.29 \text{ units}^2$$



Method II:

$$A = \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| \sin\theta$$

Using the Pythagorean identity for sine and cosine, we have: $\sin\theta = +\sqrt{1-\left(\frac{-7}{\sqrt{5}\sqrt{14}}\right)^2} = \sqrt{1-\frac{7}{10}} = \sqrt{\frac{3}{10}}$

$$A = \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| \sin\theta = \frac{1}{2}\sqrt{5}\sqrt{14}\,\frac{\sqrt{3}}{\sqrt{10}} = \frac{\sqrt{21}}{2}$$

Method III:

$$A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}\right| = \frac{1}{2}\sqrt{16+4+1} = \frac{\sqrt{21}}{2}$$

c i For l_1: $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \begin{matrix} x = 2 \\ y = -1+t, t \in \mathbb{R} \\ z = 2t \end{matrix}$

For l_2 : $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \Rightarrow \begin{matrix} x = -1 + s \\ y = 1 - 3s, s \in \mathbb{R} \\ z = 1 - 2s \end{matrix}$

ii The lines are not parallel, because the direction vectors are not parallel. Hence, we have to solve the system:

$$\begin{cases} 2 = -1 + s \Rightarrow s = 3 \\ -1 + t = 1 - 3s \\ 2t = 1 - 2s \end{cases}$$

From the first equation, $s = 3$, and substituting into the second equation: $t = 2 - 3(3) = -7$, and third equation: $2(-7) = 1 - 2(3) \Rightarrow -14 \neq -5$. Therefore, the system has no solution and the lines do not intersect.

d The shortest distance is given by $\dfrac{|(\mathbf{e} - \mathbf{d})(\mathbf{l}_1 \times \mathbf{l}_2)|}{|(\mathbf{l}_1 \times \mathbf{l}_2)|}$, where **d** and **e** are position vectors of the points on the lines, and $\mathbf{l}_1$ and $\mathbf{l}_2$ are direction vectors of the lines.

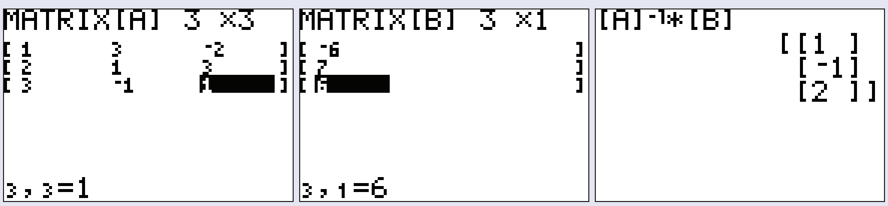
$$\mathbf{l}_1 \times \mathbf{l}_2 = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

Hence, $\dfrac{|(\mathbf{e} - \mathbf{d})(\mathbf{l}_1 \times \mathbf{l}_2)|}{|(\mathbf{l}_1 \times \mathbf{l}_2)|} = \dfrac{\left|\left(\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right)\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}\right|}{\sqrt{16 + 4 + 1}} = \dfrac{\left|\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}\right|}{\sqrt{21}} = \dfrac{9}{\sqrt{21}}$.

36 a Method I:

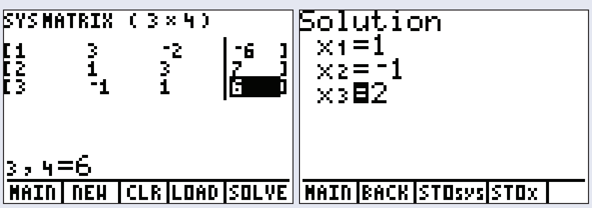
We will use matrices and their properties:

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 3 & -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -6 \\ 7 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$



Method II:

We will use PolySmlt:



b $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 2 & 1 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = \begin{pmatrix} 11 \\ -7 \\ -5 \end{pmatrix}$

c Method I:

$$\mathbf{u} = m\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + n\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} m + 2n \\ 3m + n \\ -2m + 3n \end{pmatrix}$$

$$\mathbf{vu} = \begin{pmatrix} 11 \\ -7 \\ -5 \end{pmatrix}\begin{pmatrix} m+2n \\ 3m+n \\ -2m+3n \end{pmatrix} = 11m + 22n - 21m - 7n + 10m - 15n = 0$$

Method II:

From part **b**, **v** is perpendicular to both **a** and **b**, so **va** $= 0$ and **vb** $= 0$. Hence, $\mathbf{v}(m\mathbf{a} + n\mathbf{b}) = m\mathbf{va} + n\mathbf{vb} = 0$, for all values of m and n.

d The line is perpendicular to vector **v** and to the vector $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$. So, a direction vector of the line is:

$$\begin{pmatrix} 11 \\ -7 \\ -5 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 11 & -7 & -5 \\ 3 & -1 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -7 & -5 \\ -1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 11 & -5 \\ 3 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 11 & -7 \\ 3 & -1 \end{vmatrix} = \begin{pmatrix} -12 \\ -26 \\ 10 \end{pmatrix} = -2\begin{pmatrix} 6 \\ 13 \\ -5 \end{pmatrix},$$

and a vector equation of the line is: $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ 13 \\ -5 \end{pmatrix}$

37 a i $\overrightarrow{AB} = \begin{pmatrix} 1-1 \\ 2-3 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2-1 \\ 3-3 \\ 6-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 3 \\ 1 & 0 & 5 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 3 \\ 0 & 5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$$

ii $A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|\begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}\right| = \frac{1}{2}\sqrt{25+9+1} = \frac{\sqrt{35}}{2}$

b i The plane contains the point A and its normal is $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$; hence, for the Cartesian equation, it holds:

$-5(x-1) + 3(y-3) + 1(z-1) = 0 \Rightarrow -5x + 3y + z = -5 + 9 + 1$

The equation is: $-5x + 3y + z = 5$.

ii The line contains the point D and its direction vector is $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$; hence, the Cartesian equation of the line is:

$\frac{x-5}{-5} = \frac{y+2}{3} = \frac{z-1}{1}$

c We will firstly write the equation of the line in parametric form, and then solve the system:

$x = 5 - 5t$

$y = -2 + 3t$

$z = 1 + t$

$-5(5-5t) + 3(-2+3t) + (1+t) = 5 \Rightarrow -25 + 25t - 6 + 9t + 1 + t = 5 \Rightarrow 35t = 35 \Rightarrow t = 1$

The point is: $\left.\begin{array}{l} x = 5 - 5(1) = 0 \\ y = -2 + 3(1) = 1 \\ z = 1 + (1) = 2 \end{array}\right\} \Rightarrow (0, 1, 2)$

d The distance is the same as the distance between points D and P:

$d = \sqrt{(5-0)^2 + (-2-1)^2 + (1-2)^2} = \sqrt{25+9+1} = \sqrt{35}$

38 a The line contains the point A and its direction vector is $\begin{pmatrix}1\\1\\1\end{pmatrix}$; hence, the Cartesian equation of the line is:

$$\frac{x-2}{1}=\frac{y-5}{1}=\frac{z+1}{1}$$

b We will firstly write the equation of the line in parametric form, and then solve the system:

$x=2+t$

$y=5+t$

$z=-1+t$

$$1(2+t)+1(5+t)+1(-1+t)-1=0 \Rightarrow 2+t+5+t-1+t-1=0 \Rightarrow 3t=-5 \Rightarrow t=-\frac{5}{3}$$

The point is: $\left.\begin{aligned} x&=2+\left(-\frac{5}{3}\right)\\ y&=5+\left(-\frac{5}{3}\right)\\ z&=-1+\left(-\frac{5}{3}\right)\end{aligned}\right\} \Rightarrow \left(\frac{1}{3},\frac{10}{3},-\frac{8}{3}\right)$

c Method I:

Denote the image point A'. Then, the point of intersection, $\left(\frac{1}{3},\frac{10}{3},-\frac{8}{3}\right)$, of the line and the plane is the midpoint of AA'. Hence, $\left(\frac{1}{3},\frac{10}{3},-\frac{8}{3}\right)=\left(\frac{2+x'}{2},\frac{5+y'}{2},\frac{-1+z'}{2}\right)\Rightarrow$

$\frac{1}{3}=\frac{2+x'}{2}\Rightarrow x'=\frac{2}{3}-2=-\frac{4}{3}$

$\frac{10}{3}=\frac{5+y'}{2}\Rightarrow y'=\frac{20}{3}-5=\frac{5}{3}$

$-\frac{8}{3}=\frac{-1+z'}{2}\Rightarrow z'=-\frac{16}{3}+1=-\frac{13}{3}$

Thus, $A'\left(-\frac{4}{3},\frac{5}{3},-\frac{13}{3}\right)$.

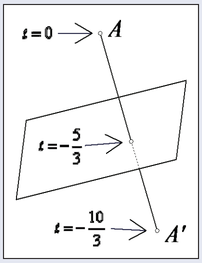
Method II:

Parameters of the points of the line are:

$t=0$ for A

$t=-\frac{5}{3}$ for the intersection; hence, $t=-\frac{10}{3}$ for the reflected point.

Thus: $\left.\begin{aligned} x'&=2+\left(-\frac{10}{3}\right)\\ y'&=5+\left(-\frac{10}{3}\right)\\ z'&=-1+\left(-\frac{10}{3}\right)\end{aligned}\right\} \Rightarrow \left(-\frac{4}{3},\frac{5}{3},-\frac{13}{3}\right)$



d We have: $\overrightarrow{AB}=\begin{pmatrix}2-2\\0-5\\6+1\end{pmatrix}=\begin{pmatrix}0\\-5\\7\end{pmatrix}$, and a direction vector of the line $\mathbf{d}=\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Hence:

$$d=\frac{\left|\overrightarrow{AB}\times\mathbf{d}\right|}{|\mathbf{d}|}=\frac{\left|\begin{pmatrix}0\\-5\\7\end{pmatrix}\times\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{1+1+1}}=\frac{\left|\begin{pmatrix}-12\\7\\5\end{pmatrix}\right|}{\sqrt{3}}=\frac{\sqrt{218}}{\sqrt{3}}\left(=\frac{\sqrt{654}}{3}\right)$$

39 a The plane contains the point P and its normal is $\begin{pmatrix}3\\-4\\1\end{pmatrix}$; hence, the Cartesian equation of the plane is:

$$3(x-1)-4(y-2)+1(z-11)=0\Rightarrow 3x-4y+z=6$$

b i $(1)+3(2)-(11)=1+6-11=4$; hence, P lies in π_2.

ii The intersection of the planes contains the point P and its direction vector is the vector product of the normals:

$\begin{pmatrix}3\\-4\\1\end{pmatrix}\times\begin{pmatrix}1\\3\\-1\end{pmatrix}=\begin{pmatrix}1\\4\\13\end{pmatrix}$. Hence, a vector equation of the line is:

$$\mathbf{r}=\begin{pmatrix}1\\2\\11\end{pmatrix}+t\begin{pmatrix}1\\4\\13\end{pmatrix},\ t\in\mathbb{R}$$

c The angle between the normals is:

$$\cos\theta_1=\frac{\begin{pmatrix}3\\-4\\1\end{pmatrix}\begin{pmatrix}1\\3\\-1\end{pmatrix}}{\sqrt{9+16+1}\sqrt{1+9+1}}=\frac{-10}{\sqrt{26\cdot 11}}$$

Hence, the angle between the planes is: $\cos\theta=\dfrac{10}{\sqrt{26\cdot 11}}\Rightarrow\theta=53.7498\ldots^\circ\approx 53.7^\circ$.

40 a $\dfrac{x+2}{3}=\dfrac{y}{1}=\dfrac{z-9}{-2}=\mu\Rightarrow x=-2+3\mu,\ y=\mu,\ z=9-2\mu$; hence: $M(-2+3\mu,\ \mu,\ 9-2\mu)$

b i $\dfrac{x-4}{3}=\dfrac{y}{1}=\dfrac{z+3}{-2}$

ii $\overrightarrow{PM}=\begin{pmatrix}-2+3\mu-4\\\mu-0\\9-2\mu+3\end{pmatrix}=\begin{pmatrix}3\mu-6\\\mu\\-2\mu+12\end{pmatrix}$

c i $\overrightarrow{PM}\cdot\begin{pmatrix}3\\1\\-2\end{pmatrix}=0\Rightarrow\begin{pmatrix}3\mu-6\\\mu\\-2\mu+12\end{pmatrix}\begin{pmatrix}3\\1\\-2\end{pmatrix}=0\Rightarrow 9\mu-18+\mu+4\mu-24=0\Rightarrow 14\mu=42\Rightarrow\mu=3$

ii The distance between the lines of magnitude of the vector $\overrightarrow{PM}$, where = 3:

$\overrightarrow{PM}=\begin{pmatrix}3\mu-6\\\mu\\-2\mu+12\end{pmatrix}=\begin{pmatrix}3\\3\\6\end{pmatrix}$; hence, the distance is: $d=\sqrt{9+9+36}=\sqrt{54}\ \left(=3\sqrt{6}\right)$

d A normal to the plane equals: $\begin{pmatrix}3\\3\\6\end{pmatrix}\times\begin{pmatrix}3\\1\\-2\end{pmatrix}=\begin{pmatrix}-12\\24\\-6\end{pmatrix}=-6\begin{pmatrix}2\\-4\\1\end{pmatrix}$; hence, the Cartesian equation of the plane is:

$$2(x-4)-4(y-0)+1(z+3)=0\Rightarrow 2x-4y+z=5$$

Solution Paper 1 type

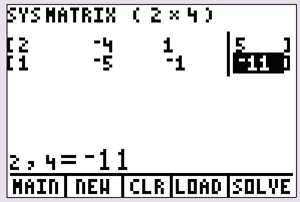
e The line is on π_1 (from part **d**).

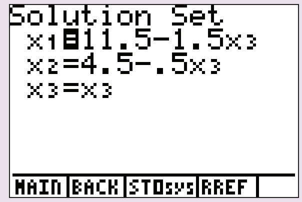
Testing the line on π_2: $(-2+3\mu)-5(\mu)-(9-2\mu)=-2+3\mu-5\mu-9+2\mu=-11$. Therefore, the line is in both planes; hence, l_1 is the line of intersection.

Solution Paper 2 type

e Solve the system:

$$\begin{cases} 2x-4y+z=5 \\ x-5y-z=-11 \end{cases}$$





So, the intersection is the line:

$$\mathbf{r}=\begin{pmatrix}11.5\\4.5\\0\end{pmatrix}+t\begin{pmatrix}-1.5\\-0.5\\1\end{pmatrix}, t\in\mathbb{R}$$

A direction vector of the line is: $-2\cdot\begin{pmatrix}-1.5\\-0.5\\1\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$, and, for $t=9$, the position vector of the point is:

$$\begin{pmatrix}11.5\\4.5\\0\end{pmatrix}+9\begin{pmatrix}-1.5\\-0.5\\1\end{pmatrix}=\begin{pmatrix}-2\\0\\9\end{pmatrix}$$, which is line l_1.

41 a L_1: $x=2+t, y=2+3t, z=3+t$

L_2: $x=2+s, y=3+4s, z=4+2s$

Hence, at the point of intersection: $\begin{cases} x=2+t=2+s \\ y=2+3t=3+4s \\ z=3+t=4+2s \end{cases}$

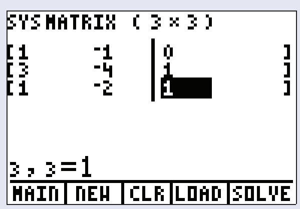
Method I:

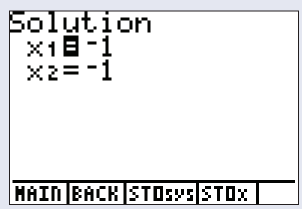
From the first equation, we have $t=s$; from the second, $2+3t=3+4t \Rightarrow t=-1$. Substituting into the third equation:

$\begin{cases} 3+(-1)=2 \\ 4+2(-1)=2 \end{cases}$; hence, the lines intersect at the point $(1,-1,2)$.

Method II:

$$\begin{cases} x=2+t=2+s \\ y=2+3t=3+4s \\ z=3+t=4+2s \end{cases} \Rightarrow \begin{cases} t-s=2-2 \\ 3t-4s=3-2 \\ t-2s=4-3 \end{cases}$$





Hence, $t=s=-1$ and the point of intersection is $(1,-1,2)$.

b The normal to the plane is perpendicular to both direction vectors, hence:

$\begin{pmatrix}1\\3\\1\end{pmatrix}\times\begin{pmatrix}1\\4\\2\end{pmatrix}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$. Since the plane contains the intersection point $(1,-1,2)$, the Cartesian equation of the plane is:

$2(x-1)-1(y+1)+1(z-2)=0 \Rightarrow 2x-y+z=5$.

c The midpoint M of $[PQ]$ is: $M = \left(\frac{1+3}{2}, \frac{-1+4}{2}, \frac{2+3}{2}\right) = \left(2, \frac{3}{2}, \frac{5}{2}\right)$.

The vector $\overrightarrow{MS}$ is parallel to the normal to the plane π, so $\overrightarrow{MS} = t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2t \\ -t \\ t \end{pmatrix}$; hence,

$S\left(2t+2, -t+\frac{3}{2}, t+\frac{5}{2}\right)$. From $\left|\overrightarrow{PS}\right| = 3 \Rightarrow$

$$\sqrt{(2t+2-1)^2 + \left(-t+\frac{3}{2}+1\right)^2 + \left(t+\frac{5}{2}-2\right)^2} = \sqrt{(2t+1)^2 + \left(-t+\frac{5}{2}\right)^2 + \left(t+\frac{1}{2}\right)^2}$$

$$\sqrt{6t^2 + \frac{15}{2}} = 3 \Rightarrow t^2 = \frac{1}{4} \Rightarrow t = \pm\frac{1}{2}$$

So, the possible solutions for S are:

$S_1\left(1+2, -\frac{1}{2}+\frac{3}{2}, \frac{1}{2}+\frac{5}{2}\right) = (3, 1, 3)$ and $S_2\left(-1+2, \frac{1}{2}+\frac{3}{2}, -\frac{1}{2}+\frac{5}{2}\right) = (1, 2, 2)$.

Note: We used the fact that $\left|\overrightarrow{PS}\right| = 3$. The line L is the symmetry line of the segment $[PQ]$; hence, $\left|\overrightarrow{QS}\right|$ should be 3.

That means that we will have the same equations if we use $\left|\overrightarrow{QS}\right| = 3$.

42 a i $L_1: \begin{aligned} x &= 2-2\lambda+\mu \\ y &= 1+\lambda-3\mu \\ z &= 1+8\lambda-9\mu \end{aligned}$ $\qquad L_2: \begin{aligned} x &= 2+s+t \\ y &= 0+2s+t \\ z &= 1+s+t \end{aligned}$

Hence, at the points of intersection: $\begin{cases} 2-2\lambda+\mu = 2+s+t \\ 1+\lambda-3\mu = 2s+t \\ 1+8\lambda-9\mu = 1+s+t \end{cases}$

Subtracting the third equation from the first, we have: $1-10\lambda+10\mu = 1 \Rightarrow \lambda = \mu$.

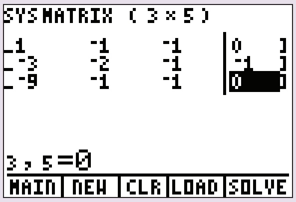
Solution Paper 1 type

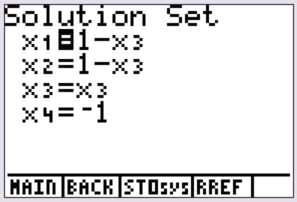
ii If $\lambda = \mu$ for points on the plane L_1, then those points are on the line: $\begin{aligned} x &= 2-2\lambda+\lambda = 2-\lambda \\ y &= 1+\lambda-3\lambda = 1-2\lambda \\ z &= 1+8\lambda-9\lambda = 1-\lambda \end{aligned}$, whose vector

equation is: $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$.

Solution Paper 2 type

ii We have to solve the system: $\begin{cases} 2-2\lambda+\mu=2+s+t \\ 1+\lambda-3\mu=2s+t \\ 1+8\lambda-9\mu=1+s+t \end{cases} \Rightarrow \begin{cases} -2\lambda+\mu-s-t=0 \\ \lambda-3\mu-2s-t=-1 \\ 8\lambda-9\mu-s-t=0 \end{cases}$





If we substitute into L_2: $s=k, t=-1$, we have: $\begin{aligned} x&=2+k-1=1+k \\ y&=0+2k-1=-1+2k \\ z&=1+k-1=k \end{aligned}$

Hence, the intersection line is: $\mathbf{r}=\begin{pmatrix}1\\-1\\0\end{pmatrix}+k\begin{pmatrix}1\\2\\1\end{pmatrix}, k\in\mathbb{R}$.

Note: If we substitute into L_1: $\lambda=1-l, \mu=1-l$, we have: $\begin{aligned} x&=2-2(1-l)+(1-l)=1+l \\ y&=1+(1-l)-3(1-l)=-1+2l \\ z&=1+8(1-l)-9(1-l)=l \end{aligned}$; hence,

$\mathbf{r}=\begin{pmatrix}1\\-1\\0\end{pmatrix}+l\begin{pmatrix}1\\2\\1\end{pmatrix}, l\in\mathbb{R}$.

Note: The lines found above are the same, since the direction vectors are parallel and, if we substitute $l=1$ into the second equation, we will have the point $\begin{pmatrix}2\\1\\1\end{pmatrix}$ from the first line. Hence, we can use any solutions in the remainder of the task.

b The point $(2, 0, -1)$ from the line is on the plane. Hence, the Cartesian equation is:
$3(x-2)-2(y)+(z+1)=0 \Rightarrow 3x-2y+z=5$

c The planes intersect at the line $\mathbf{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$, $\lambda\in\mathbb{R}$, so we have to find the intersection of this line and the plane π_3.

We will firstly write the equation of the line in parametric form, and then solve the system:

$x=2-\lambda$

$y=1-2\lambda$

$z=1-\lambda$

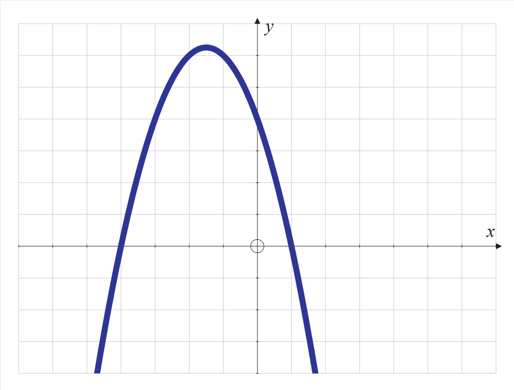
$3(2-\lambda)-2(1-2\lambda)+(1-\lambda)=5 \Rightarrow 5=5$

The equation is satisfied by any real value of λ; hence, the plane π_3 contains the line, and the intersection of the three planes is the line $\mathbf{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$, $\lambda\in\mathbb{R}$.

Chapter 15

Practice questions

1 The important points of the first derivative are zeros, where the original function reaches a minimum or maximum, $x = -4$ and $x = 1$, and the maximum point, at the midpoint. The intervals of the positive and negative values of the first derivative are to be established by the increasing/decreasing intervals of the original function. So: $f'(x) > 0, -4 < x < 1$, and $f'(x) < 0, x < -4$ or $x > 1$.



2 **a** Given the product form of the function, the values a and b are zeros that can be easily read from the graph. Therefore, **i** $a = -4$ and **ii** $b = 2$.

b **i** We can use the product rule but we have to be careful since there are three factors.

$$f'(x) = -(x+4)(x-2) - x(x-2) - x(x+4) = -\left(x^2 + 2x - 8 + x^2 - 2x + x^2 + 4x\right)$$
$$= -\left(3x^2 + 4x - 8\right) = 8 - 4x - 3x^2$$

ii $f'(x) = 0 \Rightarrow 8 - 4x - 3x^2 = 0 \Rightarrow 3x^2 + 4x - 8 = 0 \Rightarrow$

$$x = \frac{-4 \pm \sqrt{16 + 96}}{6} = \frac{-4 \pm 4\sqrt{7}}{6} = \frac{-2 \pm 2\sqrt{7}}{3} \Rightarrow$$

$$x = \frac{-2 - 2\sqrt{7}}{3} \text{ or } x = \frac{-2 + 2\sqrt{7}}{3}$$

iii $f(1) = -1(1+4)(1-2) = 5$

c **i** $m = f'(0) = 8 \Rightarrow$ Equation of tangent: $y = 8x$

ii $-x(x+4)(x-2) = 8x \Rightarrow x\left(\cancel{8} + x^2 + 2x - \cancel{8}\right) = 0 \Rightarrow x^2(x+2) = 0$. Since the point differs from the origin, we can conclude that the x-coordinate of the second point is $x = -2$.

3 **a** **i** When $t = 0: v(0) = 66 - 66e^{-0.15\times 0} = 66 - 66 \times 1 = 0$

ii When $t = 10: v(10) = 66 - 66e^{-0.15\times 10} = 66\left(1 - e^{-1.5}\right) \approx 51.3$ m/s

b **i** $a(t) = v'(t) = -66e^{-0.15t} \times (-0.15) = 9.9e^{-0.15t}$

ii $a(0) = 9.9e^{-0.15\times 0} = 9.9$ m/s^2

c **i** $\lim\limits_{t\to\infty}\left(66 - 66\overbrace{e^{-0.15t}}^{\to 0}\right) = 66$

ii $\lim\limits_{t\to\infty}\left(9.9\overbrace{e^{-0.15t}}^{\to 0}\right) = 0$

iii Since the velocity is constant (66 m/s), the acceleration must be zero.

4 a $y' = 3x^2 + 14x + 8 \Rightarrow y' = 0 \Rightarrow 3x^2 + 14x + 8 = 0 \Rightarrow (3x+2)(x+4) = 0 \Rightarrow x = -\frac{2}{3}$ or $x = -4$

x	$x < -4$	-4	$-4 < x < -\frac{2}{3}$	$-\frac{2}{3}$	$x > -\frac{2}{3}$
$f'(x)$	positive	0	negative	0	positive
$f(x)$	increases	maximum	decreases	minimum	increases

To find the exact coordinates, we can use synthetic division, also known as Horner's algorithm.

$f(x)/x$	1	7	8	-3
-4	1	3	-4	13
$-\frac{2}{3}$	1	$\frac{19}{3}$	$\frac{34}{9}$	$-\frac{149}{27}$

So, the maximum point is $(-4, 13)$ and the minimum point is $\left(-\frac{2}{3}, -\frac{149}{27}\right)$.

b $y'' = 6x + 14 \Rightarrow 6x + 14 = 0 \Rightarrow x = -\frac{14}{6} = -\frac{7}{3}$

$$f\left(-\frac{7}{3}\right) = \left(-\frac{7}{3}\right)^3 + 7\times\left(-\frac{7}{3}\right)^2 + 8\times\left(-\frac{7}{3}\right) - 3 = -\frac{343}{27} + \frac{343}{9} - \frac{56}{3} - 3 = \frac{-343 + 1029 - 504 - 81}{27} = \frac{101}{27}$$

So, the point of inflexion is $\left(-\frac{7}{3}, \frac{101}{27}\right)$.

Note: Horner's algorithm could also be used here.

$f(x)/x$	1	7	8	-3
$-\frac{7}{3}$	1	$\frac{14}{3}$	$-\frac{26}{9}$	$\frac{101}{27}$

5 a i $g(x) = 2 + e^{-3x} \Rightarrow g'(x) = e^{-3x} \times -3 = -3e^{-3x}$

ii Since $-3e^{-3x} < 0$ for all real values of x, we can conclude that the function always decreases.

b i $g(x) = 2 + e^{-3x} \Rightarrow g\left(-\frac{1}{3}\right) = 2 + e^{-3\times\left(-\frac{1}{3}\right)} = 2 + e$

ii $g'(x) = -3e^{-3x} \Rightarrow g'\left(-\frac{1}{3}\right) = -3e^{-3\times\left(-\frac{1}{3}\right)} = -3e$

c $y = -3e\left(x + \frac{1}{3}\right) + 2 + e \Rightarrow y = -3ex \cancel{-e} + 2 \cancel{+e} = -3ex + 2$

6 a Firstly, we are going to write the function in product form:

$$f(x) = \left(2x^2 - 13x + 20\right)(x-1)^{-2} \Rightarrow f'(x) = (4x-13)(x-1)^{-2} + \left(2x^2 - 13x + 20\right)\times(-2)(x-1)^{-3}$$

$$= (x-1)^{-3}\left((4x-13)(x-1) + \left(2x^2 - 13x + 20\right)\times(-2)\right) = \frac{\cancel{4x^2} - 17x + 13 \cancel{-4x^2} + 26x - 40}{(x-1)^3}$$

$$= \frac{9x - 27}{(x-1)^3}, x \neq 1$$

b We know that a minimum point has the first derivative equal to zero, and therefore:

$f'(x) = 0 \Rightarrow \dfrac{9x - 27}{(x-1)^3} = 0 \Rightarrow 9x - 27 = 0 \Rightarrow x = \dfrac{27}{9} = 3$. Also, the second derivative must be positive, so:

$$f''(x) = \frac{72 - 18x}{(x-1)^4} \Rightarrow f''(3) = \frac{72 - 18 \times 3}{(3-1)^4} = \frac{72 - 54}{16} = \frac{18}{16} = \frac{9}{8} > 0.$$

Therefore, $f(3)$ is a minimum.

c For the point of inflexion, the second derivative must be equal to zero.

$$f''(x) = 0 \Rightarrow \frac{72 - 18x}{(x-1)^4} = 0 \Rightarrow 72 - 18x = 0 \Rightarrow x = \frac{72}{18} = 4$$

$$y = f(4) = \frac{2 \times 4^2 - 13 \times 4 + 20}{(4-1)^2} = 0 \Rightarrow I(4, 0)$$

7 a Rewriting as an expression having an integer exponent:

$$y = (2x+3)^{-2} \Rightarrow y' = -2 \times (2x+3)^{-3} \times 2 = \frac{-4}{(2x+3)^3},\ x \neq -\frac{3}{2}$$

b $y = e^{\sin(5x)} \Rightarrow y' = e^{\sin(5x)} \times \cos(5x) \times 5 = 5\cos(5x)\, e^{\sin(5x)}$

c $y = \tan^2\left(x^2\right) \Rightarrow y' = 2\tan\left(x^2\right)\sec^2\left(x^2\right) \times 2x = 4x\tan\left(x^2\right)\sec^2\left(x^2\right)$ or $y' = \dfrac{4x\sin\left(x^2\right)}{\cos^3\left(x^2\right)}$

8 $y = Ax + B + \dfrac{C}{x} \Rightarrow y' = A - \dfrac{C}{x^2} \Rightarrow y' = 0 \Rightarrow A = \dfrac{C}{x^2} \Rightarrow x^2 = \dfrac{C}{A} \Rightarrow x = \pm\sqrt{\dfrac{C}{A}}$

By observing the given stationary points, we can establish a relationship between A and C: $x = \pm 1 \Rightarrow \dfrac{C}{A} = 1 \Rightarrow C = A$.

Now, we need to use the fact that points P and Q lie on the curve itself, and therefore their coordinates satisfy the equation of the curve.

$$\left.\begin{array}{l} P(1,4) \Rightarrow 4 = A \times 1 + B + \dfrac{A}{1} \Rightarrow 2A + B = 4 \\ Q(-1,0) \Rightarrow 0 = A \times (-1) + B + \dfrac{A}{-1} \Rightarrow -2A + B = 0 \end{array}\right\} \Rightarrow 2B = 4 \Rightarrow B = 2 \Rightarrow A = C = 1$$

9 $x^3 + y^3 = 2 \Big/ \dfrac{d}{dx} \Rightarrow 3x^2 + 3y^2 y' = 0 \Rightarrow y' = -\dfrac{\cancel{3}x^2}{\cancel{3}y^2} = -\dfrac{x^2}{y^2} \Rightarrow y'(1,1) = -\dfrac{1^2}{1^2} = -1$

$3x^2 = -3y^2 y' \Big/ \dfrac{d}{dx} \Rightarrow \overbrace{\cancel{6}}^{2} x = -\cancel{3}\left(2yy' \times y' + y^2 y''\right) \Rightarrow 2x + 2y\left(y'\right)^2 = -y^2 y''$

$$y'' = -\frac{2x + 2y\left(y'\right)^2}{y^2} \Rightarrow y''(1,1) = -\frac{2 \times 1 + 2 \times 1 \times (-1)^2}{1^2} = -4$$

10 a $y = \dfrac{x}{e^x - 1} \Rightarrow y' = \dfrac{1 \times \left(e^x - 1\right) - x \times e^x}{\left(e^x - 1\right)^2} = \dfrac{(1 - x)e^x - 1}{\left(e^x - 1\right)^2}$

b $y = e^x \sin 2x \Rightarrow y' = e^x \sin 2x + e^x \cos 2x \times 2 = e^x(\sin 2x + 2\cos 2x)$

c $y = \left(x^2 - 1\right)\ln 3x \Rightarrow y' = 2x \times \ln 3x + \left(x^2 - 1\right) \times \dfrac{1}{x} = 2x\ln 3x + x - \dfrac{1}{x}$

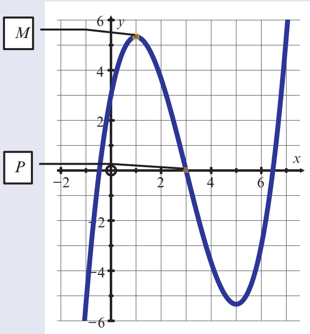
11 $y = x^2 - 4x \Rightarrow y' = 2x - 4 \Rightarrow m_N = -\dfrac{1}{y'(3)} = -\dfrac{1}{2}$, equation of normal: $y = -\dfrac{1}{2}(x-3) - 3 \Rightarrow y = -\dfrac{1}{2}x - \dfrac{3}{2}$

$x = 0 \Rightarrow y = -\dfrac{1}{2} \times 0 - \dfrac{3}{2} = -\dfrac{3}{2}, P\left(0, -\dfrac{3}{2}\right); y = 0 \Rightarrow 0 = -\dfrac{1}{2}x - \dfrac{3}{2} \Rightarrow x = -3, Q(-3, 0)$

12 a The x-coordinate of P can be read from both graphs. In the first derivative graph, we can see that the curve has a minimum point at the point where $x = 3$, whilst, in the second graph, we can see that the line has a zero at the point where $x = 3$.

b Since point M is a maximum point, we know that the first derivative must be equal to zero; therefore, by observing the first graph, we have two possible values: $x = 1$ or $x = 5$. Now, by looking at the second graph, we can see that $x = 1 \Rightarrow y''(1) < 0$ and we have a maximum point. For the second point, where $x = 5 \Rightarrow y''(5) > 0$, we have a local minimum.

c



13 $x^2 + xy + y^2 - 3y = 10 \Big/ \frac{d}{dx} \Rightarrow 2x + y + xy' + 2yy' - 3y' = 0 \Rightarrow 2x + y = y'(3 - x - 2y) \Rightarrow y' = \frac{2x + y}{3 - x - 2y}$

$m_N = -\frac{1}{y'(2,3)} = -\frac{3 - 2 - 2 \times 3}{2 \times 2 + 3} = \frac{5}{7} \Rightarrow$ Equation of normal: $y = \frac{5}{7}(x - 2) + 3 \Rightarrow y = \frac{5}{7}x + \frac{11}{7}$

Solution Paper 1 type

14 $V = r^2\pi h \Rightarrow 128\cancel{\pi} = r^2\cancel{\pi}h \Rightarrow h = \frac{128}{r^2}$

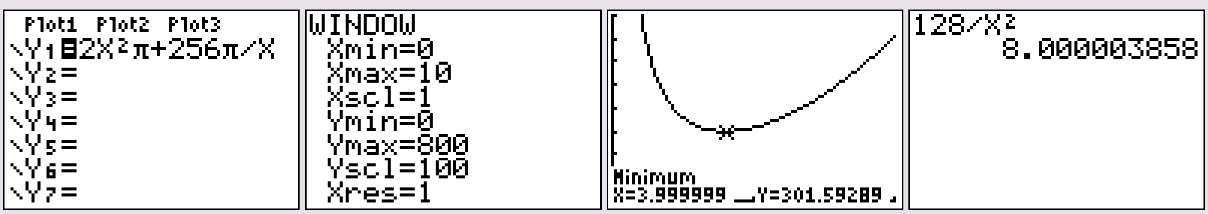
$S = 2r^2\pi + 2r\pi h \Rightarrow S(r) = 2r^2\pi + 2\cancel{r}\pi\frac{128}{r^{\cancel{2}}} = 2r^2\pi + \frac{256\pi}{r}$

$S'(r) = 4r\pi - \frac{256\pi}{r^2} = 4\pi\left(r - \frac{64}{r^2}\right) \Rightarrow S'(r) = 0 \Rightarrow r - \frac{64}{r^2} = 0 \Rightarrow r = \frac{64}{r^2} \Rightarrow r^3 = 64 \Rightarrow r = \sqrt[3]{64} = 4 \Rightarrow h = \frac{128}{4^2} = 8$

So, the radius is 4 cm and the height is 8 cm.

Solution Paper 2 type

14 On a GDC, we simply input the surface area function and calculate a minimum.



So, the minimum surface area occurs when the radius is 4 cm and the height is 8 cm.

15 Let's focus on the vertex of the rectangle in the first quadrant. The coordinates are $(x, y) = (x, 12 - x^2)$. Now, the dimensions of the rectangle and its area are: $l = 2x, w = y \Rightarrow A = lw = 2xy \Rightarrow A(x) = 2x(12 - x^2) = 24x - 2x^3$.

To find the maximum possible area, we need to find the zero of the first derivative.

$A'(x) = 24 - 6x^2 \Rightarrow A'(x) = 0 \Rightarrow 24 - 6x^2 = 0 \Rightarrow 24 = 6x^2 \Rightarrow x^2 = 4 \Rightarrow x = 2$, since the vertex is in the first quadrant. Again, we can verify that we have a maximum point since the second derivative test gives us a negative value: $A''(x) = -12x \Rightarrow A''(2) = -12 \times 2 = -24 < 0$.

So, the dimensions are $l = 2 \times 2 = 4$ and $w = 12 - 2^2 = 8$, and that gives us the maximum possible area of 32.

16 a The first derivative is negative when the function is decreasing, whilst the second derivative is negative when the function is concave down. By looking for these features, we identify point *E*.

b The first derivative is negative when the function is decreasing, whilst the second derivative is positive when the function is concave up. By looking for these features, we identify point *A*.

c The first derivative is positive when the function is increasing, whilst the second derivative is negative when the function is concave down. By looking for these features, we identify point *C*.

17 $y=\frac{2x-1}{x+2}\Rightarrow y'=\frac{2(x+2)-(2x-1)\times 1}{(x+2)^2}=\frac{\cancel{2x}+4-\cancel{2x}+1}{(x+2)^2}=\frac{5}{(x+2)^2}$

$y'(-3)=\frac{5}{(-3+2)^2}=5\Rightarrow m_n=-\frac{1}{5}$

Equation of normal: $y=-\frac{1}{5}(x+3)+7\Rightarrow y=-\frac{1}{5}x-\frac{3}{5}+7\Rightarrow y=-\frac{1}{5}x+\frac{32}{5}$

18 $y=\ln(4x-3)\Rightarrow y'=\frac{4}{4x-3}\Rightarrow y'(1)=\frac{4}{4\times 1-3}=4$

a $m_T=y'(1)=4\Rightarrow$ Equation of tangent: $y=4(x-1)+0\Rightarrow y=4x-4$

b $m_N=-\frac{1}{y'(1)}=-\frac{1}{4}\Rightarrow$ Equation of normal: $y=-\frac{1}{4}(x-1)+0\Rightarrow y=-\frac{1}{4}x+\frac{1}{4}$

19 $y=x^2\ln x\Rightarrow y'=2x\ln x+x^{\cancel{2}}\times\frac{1}{\cancel{x}}=x(2\ln x+1)\Rightarrow y''=2\ln x+1+\cancel{x}\times\frac{2}{\cancel{x}}=2\ln x+3$

a $y'=0\Rightarrow x(2\ln x+1)=0\Rightarrow 2\ln x+1=0\Rightarrow \ln x=-\frac{1}{2}\Rightarrow x=e^{-\frac{1}{2}}=\frac{1}{e^{\frac{1}{2}}}$ or $\frac{1}{\sqrt{e}}$

$y=\left(e^{-\frac{1}{2}}\right)^2\ln e^{-\frac{1}{2}}=-\frac{1}{2e}$; so, $P\left(\frac{1}{\sqrt{e}},-\frac{1}{2e}\right)$

Notice that the domain of the function is $x>0$; therefore, we discard the solution $x=0$.

$y''\left(e^{-\frac{1}{2}}\right)=2\ln e^{-\frac{1}{2}}+3=-1+3=2>0$; therefore, the point *P* is a minimum.

$\lim\limits_{x\to 0^-}y=\lim\limits_{x\to 0^-}x^2\ln x=0$ and $\lim\limits_{x\to\infty}y=\lim\limits_{x\to\infty}x^2\ln x=\infty$; therefore, the point *P* is the absolute minimum.

b $y''=0\Rightarrow 2\ln x+3=0\Rightarrow x=e^{-\frac{3}{2}}=\frac{1}{e^{\frac{3}{2}}}=\frac{1}{e\sqrt{e}}$, $y=\left(e^{-\frac{3}{2}}\right)^2\ln e^{-\frac{3}{2}}=-\frac{3}{2e^3}$; so, $I\left(\frac{1}{e\sqrt{e}},-\frac{3}{2e^3}\right)$

20 a $f(x)=x^2+\frac{a}{x}\Rightarrow f'(x)=2x-\frac{a}{x^2}\Rightarrow f''(x)=2+\frac{2a}{x^3}$

i $f'(2)=0\Rightarrow 2\times 2-\frac{a}{2^2}=0\Rightarrow a=16$

ii $f'(-3)=0\Rightarrow 2\times(-3)-\frac{a}{(-3)^2}=0\Rightarrow a=-54$

b $f'(x)=2x-\frac{a}{x^2}=0\Rightarrow 2x^3=a\Rightarrow x=\sqrt[3]{\frac{a}{2}}$, $f''\left(\sqrt[3]{\frac{a}{2}}\right)=2+\frac{2a}{\left(\sqrt[3]{\frac{a}{2}}\right)^3}=2+\frac{2\cancel{a}}{\frac{\cancel{a}}{2}}=6>0$

Since the second derivative is always positive, the stationary point cannot be a maximum.

21 A line $y=mx+l$ that passes through (3, 2) satisfies the following equation:

$2=m\times 3+l\Rightarrow l=2-3m;\ y=mx+2-3m$

$x=0\Rightarrow y=2-3m;\ y=0\Rightarrow 0=mx+2-3m\Rightarrow x=\frac{3m-2}{m}$

Therefore, the area of the triangle is given by the expression:

$A(m)=\frac{1}{2}(2-3m)\frac{3m-2}{m}=-\frac{(3m-2)^2}{2m}\Rightarrow A'(m)=-\frac{2(3m-2)\times 3\times 2m-(3m-2)^2\times 2}{4m^2}$

$=-\frac{2(3m-2)(6m-3m+2)}{4m^2}=-\frac{2(3m-2)(3m+2)}{4m^2}\Rightarrow A'(m)=0\Rightarrow(3m-2)(3m+2)=0\Rightarrow m=\frac{2}{3}$ or $m=-\frac{2}{3}$

We can discard the first solution since, for $m = \frac{2}{3}$, the line passes through the origin and so the triangle doesn't exist.

$$m = -\frac{2}{3} \Rightarrow l = 2 - 3 \times \left(-\frac{2}{3}\right);\ y = -\frac{2}{3}x + 4$$

22 $y = x\tan x$: $x = \frac{\pi}{4} \Rightarrow y = \frac{\pi}{4}\tan\frac{\pi}{4} = \frac{\pi}{4}$, $P\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

$$y'(x) = \tan x + x\sec^2 x \Rightarrow y'\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} + \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right) = 1 + \frac{\pi}{\cancel{4}\,2} \times \cancel{2} = 1 + \frac{\pi}{2}$$

$$m_T = y'\left(\frac{\pi}{4}\right) = 1 + \frac{\pi}{2} \Rightarrow \text{Equation of tangent: } y = \left(1 + \frac{\pi}{2}\right)\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4} \Rightarrow y = \left(1 + \frac{\pi}{2}\right)x - \frac{\pi^2}{8}$$

$$m_N = -\frac{1}{y'\left(\frac{\pi}{4}\right)} = -\frac{1}{1 + \frac{\pi}{2}} = -\frac{2}{2 + \pi} \Rightarrow \text{Equation of normal: } y = \left(-\frac{2}{2 + \pi}\right)\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4} \Rightarrow$$

$$y = \left(-\frac{2}{2 + \pi}\right)x + \frac{2\pi}{4(2 + \pi)} + \frac{\pi}{4} \Rightarrow y = \left(-\frac{2}{2 + \pi}\right)x + \frac{4\pi + \pi^2}{4(2 + \pi)}$$

23 a $f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \Rightarrow f'(x) = \frac{e^{-\frac{x^2}{2}} \times \frac{-\cancel{2}x}{\cancel{2}}}{\sqrt{2\pi}} = -\frac{xe^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \Rightarrow$

$$f''(x) = -\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} - x\frac{e^{-\frac{x^2}{2}} \times \frac{-\cancel{2}x}{\cancel{2}}}{\sqrt{2\pi}} = \frac{e^{-\frac{x^2}{2}}(x^2 - 1)}{\sqrt{2\pi}}$$

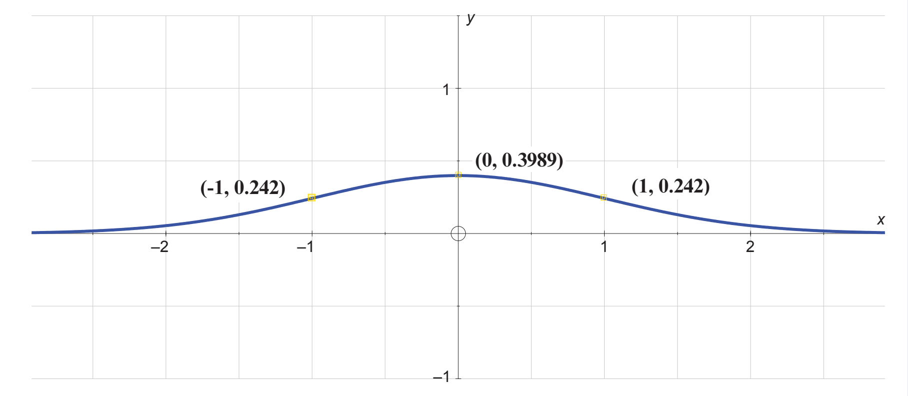
$$f'(x) = 0 \Rightarrow x = 0,\ y = f(0) = \frac{e^0}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \Rightarrow M\left(0, \frac{1}{\sqrt{2\pi}}\right)$$

$$f''(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1,\ y = -\frac{e^{-\frac{(\pm 1)^2}{2}}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2e\pi}} \Rightarrow I_1\left(-1, \frac{1}{\sqrt{2e\pi}}\right), I_2\left(1, \frac{1}{\sqrt{2e\pi}}\right)$$

b We notice that the function f is even, and therefore symmetrical with respect to the y-axis.

$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\overbrace{e^{-\frac{x^2}{2}}}^{0}}{\sqrt{2\pi}} = 0$; therefore, the x-axis is the horizontal asymptote.

c



24 a $f(x) = 2\ln(x^2 + 3) - x \Rightarrow f'(x) = 2\frac{2x}{x^2 + 3} - 1 = \frac{4x}{x^2 + 3} - 1 \left[= \frac{-x^2 + 4x - 3}{x^2 + 3}\right] \Rightarrow$

$$f'(x) = 0 \Rightarrow \frac{4x}{x^2 + 3} - 1 = 0 \Rightarrow \frac{4x}{x^2 + 3} = 1 \Rightarrow 4x = x^2 + 3 \Rightarrow 0 = x^2 - 4x + 3 \Rightarrow (x - 3)(x - 1) = 0$$

$x = 3$, $y = 2\ln 12 - 3$; or $x = 1$, $y = 2\ln 4 - 1 = 4\ln 2 - 1$

We notice that the denominator is always positive; therefore, the sign of the first derivative depends on the numerator, which is a quadratic expression that has a negative leading coefficient.

x	$]-\infty, 1[$	1	$]1, 3[$	3	$]3, \infty[$
$f'(x)$	negative	0	positive	0	negative
$f(x)$	decreasing	minimum	increasing	maximum	decreasing

To conclude: $(1, 4\ln 2 - 1)$ is a minimum point and $(3, 2\ln 12 - 3)$ is a maximum point.

b $f''(x) = \dfrac{4(x^2+3) - 4x \times 2x}{(x^2+3)^2} = \dfrac{12-4x^2}{(x^2+3)^2} \Rightarrow f''(x) = 0 \Rightarrow 12 - 4x^2 = 0 \Rightarrow x = \pm\sqrt{3}$

Again, since the denominator is always positive, we can conclude that the sign depends on the numerator only, and the numerator is a quadratic expression which changes its sign at the zeros. Therefore, we can conclude that the x-coordinates we found are those of inflexion points.

25 $f(x) = \dfrac{2x}{18+0.015x^2} \Rightarrow f'(x) = \dfrac{2(18+0.015x^2) - 2x \times 0.3x}{(18+0.015x^2)^2} = \dfrac{2(18-0.015x^2)}{(18+0.015x^2)^2}$

$f'(x) = 0 \Rightarrow 18 - 0.015x^2 = 0 \Rightarrow x^2 = \dfrac{18}{0.015} = 1200 \Rightarrow x = \sqrt{1200} = 20\sqrt{3} \approx 34.6$ km/hr

26 $2x^2 - 3y^2 = 2 \Big/ \dfrac{d}{dx} \Rightarrow 4x - 6yy' = 0 \Rightarrow y' = \dfrac{\cancel{4}2x}{\cancel{6}3y} = \dfrac{2x}{3y}$

$x = 5 \Rightarrow 2 \times 5^2 - 3y^2 = 2 \Rightarrow 48 = 3y^2 \Rightarrow y^2 = 16 \Rightarrow y = \pm\sqrt{16} = \pm 4$

When $x = 5$ we have two values of y; therefore, there are two points and hence two gradients.

$y'(5, -4) = \dfrac{\cancel{2} \times 5}{3 \times (-\cancel{4}2)} = -\dfrac{5}{6}, \quad y'(5, 4) = \dfrac{\cancel{2} \times 5}{3 \times \cancel{4}2} = \dfrac{5}{6}$

27 $y = \arccos(1-2x^2) \Rightarrow y' = \dfrac{-1}{\sqrt{1-(1-2x^2)^2}} \times (-4x) = \dfrac{4x}{\sqrt{\cancel{1} - (\cancel{1} - 4x^2 + 4x^4)}}$

$$= \frac{\cancel{4}2x}{\cancel{2}|x|\sqrt{1-x^2}} = \begin{cases} \dfrac{-2}{\sqrt{1-x^2}}, & -1 < x < 0 \\ \dfrac{2}{\sqrt{1-x^2}}, & 0 \leqslant x < 1 \end{cases} = \frac{2 \times \operatorname{sign}(x)}{\sqrt{1-x^2}}$$

Note: We had to restrict the domain of the derivative because of the expression in the denominator.

28 $f(x) = x^2 \ln x \Rightarrow f'(x) = 2x\ln x + x^{\cancel{2}} \times \dfrac{1}{\cancel{x}} = x(2\ln x + 1), x > 0$

29 $f(x) = \dfrac{1}{2}\sin 2x + \cos x \Rightarrow f'(x) = \dfrac{1}{\cancel{2}}\cos 2x \times \cancel{2} - \sin x = 1 - 2\sin^2 x - \sin x = (1 - 2\sin x)(1 + \sin x)$

$f'(x) = 0 \Rightarrow (1 - 2\sin x)(1 + \sin x) = 0 \Rightarrow \sin x = \dfrac{1}{2}$ or $\sin x = -1$

30 $3x^2 + 4y^2 = 7 \Big/ \dfrac{d}{dx} \Rightarrow 6x + 8yy' = 0 \Rightarrow y' = -\dfrac{\cancel{6}3x}{\cancel{8}4y} = -\dfrac{3x}{4y}$

$x = 1 \Rightarrow 3 \times 1^2 + 4y^2 = 7 \Rightarrow 4y^2 = 4 \Rightarrow y^2 = 1 \Rightarrow y = 1$ because $y > 0$.

$y'(1, 1) = -\dfrac{3 \times 1}{4 \times 1} = -\dfrac{3}{4}$

31 a $f(x) = \ln(2x-1) \Rightarrow f'(x) = \dfrac{2}{2x-1}$

Solution Paper 1 type

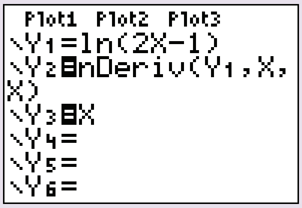
b $f'(x) = x \Rightarrow \dfrac{2}{2x-1} = x \Rightarrow 2 = 2x^2 - x \Rightarrow 0 = 2x^2 - x - 2 \Rightarrow x_{1,2} = \dfrac{1 \pm \sqrt{1+16}}{4} = \dfrac{1 \pm \sqrt{17}}{4}$

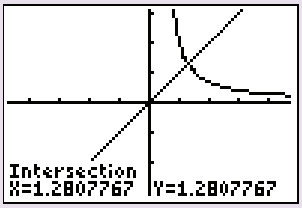
We need to check whether both solutions are in the domain of the function *f*. We notice that the domain is:

$D(f) = x\left\{x \in \mathbb{R} \middle| x > \dfrac{1}{2}\right\}$; therefore, there is only one solution: $x = \dfrac{1+\sqrt{17}}{4} \approx 1.28$ (correct to three significant figures).

Solution Paper 2 type

b We can draw the derivative function on a calculator without finding the formula for it. To obtain a clearer diagram, we deselect the original function so that we can find the point of intersection easier with the identity function.





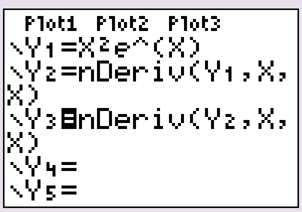
Solution Paper 1 type

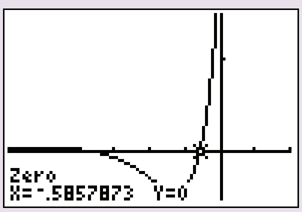
32 $f(x) = x^2 e^x \Rightarrow f'(x) = 2xe^x + x^2 e^x \Rightarrow f''(x) = 2e^x + 2xe^x + 2xe^x + x^2 e^x = e^x\left(2 + 4x + x^2\right)$

Since $e^x > 0 : f''(x) = 0 \Rightarrow x^2 + 4x + 2 = 0 \Rightarrow x_{1,2} = -2 \pm \sqrt{4-2} = -2 \pm \sqrt{2}$. The condition of the question is that the *x*-coordinate of the point of inflexion is between −2 and 0; therefore, there is only one solution: $x = -2 + \sqrt{2} \approx -0.586$.

Solution Paper 2 type

32 As in the previous question, we can draw the graph of the second derivative and then simply find the zero that is between −2 and 0.





33 Since the normal has a slope of −2, we need to find a point with a gradient of $\dfrac{1}{2}$.

$y = \arctan(x-1) \Rightarrow y' = \dfrac{1}{1+(x-1)^2} = \dfrac{1}{2 - 2x + x^2}$

$y'(x) = \dfrac{1}{2} \Rightarrow 2 - 2x + x^2 = 2 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 2$ given the condition $x > 0$.

$x = 2 \Rightarrow y = \arctan(1) = \dfrac{\pi}{4} \Rightarrow \dfrac{\pi}{4} = -2 \times 2 + c \Rightarrow c = \dfrac{\pi}{4} + 4 \approx 4.79$ (correct to three significant figures)

34 a $f(x) = e^{1+\sin \pi x} \Rightarrow f'(x) = e^{1+\sin \pi x} \times \cos(\pi x) \times \pi = \pi \cos(\pi x)\, e^{1+\sin \pi x}$

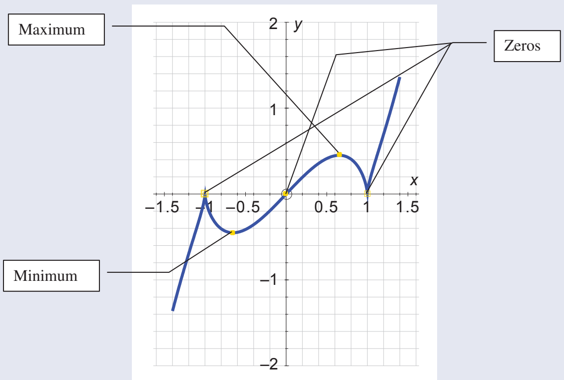
b Since $x \geqslant 0$, we need to find the x-coordinates of the points where $f\,'(x) = 0$:

$$\cos(\pi x) = 0 \Rightarrow \pi x = \frac{\pi}{2} + n\pi \Rightarrow x = \frac{1}{2} + n,\, n \in \mathbb{N}$$

$$x_0 = \frac{1}{2},\, x_1 = \frac{3}{2},\, x_2 = \frac{5}{2},\, \ldots,\, x_n = \frac{2n+1}{2},\, n \in \mathbb{N}$$

35 a $f(x) = x\sqrt[3]{\left(x^2-1\right)^2},\ -1.4 \leqslant x \leqslant 1.4$

In this instance, we need to set up the window on the calculator properly and simply copy it onto graphic paper.



b i $f(x) = x\sqrt[3]{\left(x^2-1\right)^2} \Rightarrow f\,'(x) = \sqrt[3]{\left(x^2-1\right)^2} + x \times \frac{2}{3}\left(x^2-1\right)^{-\frac{1}{3}} \times 2x = \frac{3x^2-3+4x^2}{3\sqrt[3]{x^2-1}} = \frac{7x^2-3}{3\sqrt[3]{x^2-1}}$

$D\left(f^{-1}\right) = \{x \in \mathbb{R} \mid -1.4 \leqslant x \leqslant 1.4\}$

ii $f\,'(x) = 0 \Rightarrow \frac{7x^2-3}{3\sqrt[3]{x^2-1}} = 0 \Rightarrow 7x^2 - 3 = 0 \Rightarrow x_{1,2} = \pm\sqrt{\frac{3}{7}} = \frac{\pm\sqrt{21}}{7}$

Therefore, the minimum occurs for $x = -\sqrt{\frac{3}{7}}$, whilst the maximum occurs for $x = \sqrt{\frac{3}{7}}$.

c For ease of calculation of the second derivative, we will rewrite the first derivative as a product.

$$f\,'(x) = \frac{1}{3}\left(7x^2-3\right)\left(x^2-1\right)^{-\frac{1}{3}} \Rightarrow f\,''(x) = \frac{1}{3}\left[14x\left(x^2-1\right)^{-\frac{1}{3}} + \left(7x^2-3\right)\times\left(-\frac{1}{2}\right)\left(x^2-1\right)^{-\frac{4}{3}} \times 2x\right]$$

$$= \frac{2}{9}\left(x^2-1\right)^{-\frac{4}{3}}\left[21x\left(x^2-1\right) - \left(7x^2-3\right)\times x\right] = \frac{2}{9\left(x^2-1\right)^{\frac{4}{3}}}\left[21x^3 - 21x - 7x^3 + 3x\right] = \frac{4\left(7x^3-9x\right)}{9\left(x^2-1\right)^{\frac{4}{3}}}$$

$$f\,''(x) = 0 \Rightarrow 7x^3 - 9x = 0 \Rightarrow x = \frac{3\sqrt{7}}{7} \approx 1.1339, \text{ since } x > 0.$$

36 Given that the line $y = 16x - 9$ is tangent at (1, 7), we can conclude that $y\,'(1) = 16$. The given point on the curve gives us the second equation.

$$y = 2x^3 + ax^2 + bx - 9 \Rightarrow y\,' = 6x^2 + 2ax + b$$

$$y\,'(1) = 6 + 2a + b = 16 \Rightarrow 2a + b = 10$$

$y(1) = 2 + a + b - 9 = 7 \Rightarrow a + b = 14$, so we have a pair of simultaneous equations to solve.

$$\begin{cases} 2a + b = 10 \\ a + b = 14 \end{cases} \Rightarrow \begin{cases} b = 10 - 2a \\ a + 10 - 2a = 14 \end{cases} \Rightarrow \begin{cases} b = 10 - 2a \\ -4 = a \end{cases} \Rightarrow \begin{cases} b = 18 \\ a = -4 \end{cases}$$

37 a $y = \tan x - 8\sin x \Rightarrow \frac{dy}{dx} = \sec^2 x - 8\cos x$

b $\sec^2 x - 8\cos x = 0 \Rightarrow \frac{1 - 8\cos^3 x}{\cos^2 x} = 0 \Rightarrow 1 - 8\cos^3 x = 0 \Rightarrow \cos^3 x = \frac{1}{8} \Rightarrow \cos x = \frac{1}{2}$

38 a $y = x^3 + 4x^2 + x - 6 \Rightarrow y' = 3x^2 + 8x + 1 \Rightarrow y'(-1) = 3 - 8 + 1 = -4$

$x = -1 \Rightarrow y = -1 + 4 - 1 - 6 = -4$; equation of tangent: $y = -4(x+1) - 4 \Rightarrow y = -4x - 8$

b To find where the tangent meets the curve again, we need to solve a pair of simultaneous equations. We will use the substitution method.

$$\begin{cases} y = x^3 + 4x^2 + x - 6 \\ y = -4x - 8 \end{cases} \Rightarrow \begin{cases} -4x - 8 = x^3 + 4x^2 + x - 6 \\ y = -4x - 8 \end{cases} \Rightarrow \begin{cases} x^3 + 4x^2 + 5x + 2 = 0 \\ y = -4x - 8 \end{cases} \Rightarrow$$

$$\begin{cases} (x^3 + 4x^2 + 4x) + (x+2) = 0 \\ y = -4x - 8 \end{cases} \Rightarrow \begin{cases} x(x+2)^2 + (x+2) = 0 \\ y = -4x - 8 \end{cases} \Rightarrow \begin{cases} (x+2)(x^2 + 2x + 1) = 0 \\ y = -4x - 8 \end{cases} \Rightarrow$$

$$\begin{cases} (x+2)(x+1)^2 = 0 \\ y = -4x - 8 \end{cases} \Rightarrow \begin{cases} x = -2 \text{ or } x = -1 \\ y = -4 \times (-2) - 8 = 0 \text{ or } y = -4 \end{cases}$$

So, the second point is (–2, 0).

39 $y = \sin(kx) - kx\cos(kx) \Rightarrow y' = \cancel{\cos(kx) \times k}\ \cancel{-k\cos(kx)} + kx\sin(kx) \times k = k^2 x \sin(kx)$

40 $xy^3 + 2x^2y = 3 \Big/ \dfrac{d}{dx} \Rightarrow y^3 + x \times 3y^2y' + 4xy + 2x^2 \times y' = 0 \Rightarrow y'(3xy^2 + 2x^2) = -(y^3 + 4xy)$

$\Rightarrow y' = -\dfrac{y^3 + 4xy}{3xy^2 + 2x^2}$, $y'(1, 1) = -\dfrac{1+4}{3+2} = -1 \Rightarrow$ Equation of tangent: $y = -1(x-1) + 1 \Rightarrow y = -x + 2$

41 a i $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1} \Rightarrow f'(x) = \dfrac{(2x-1)(x^2 + x + 1) - (x^2 - x + 1)(2x+1)}{(x^2 + x + 1)^2}$

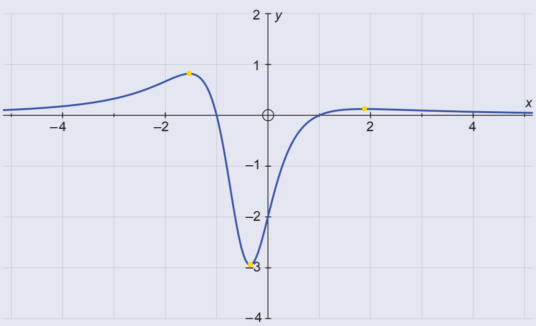
$$= \frac{\cancel{2x^3} + x^2 \cancel{+x} - 1 \cancel{-2x^3} + x^2 \cancel{-x} - 1}{(x^2 + x + 1)^2} = \frac{2x^2 - 2}{(x^2 + x + 1)^2} = \frac{2(x^2 - 1)}{(x^2 + x + 1)^2}$$

ii If the tangents are parallel to the x-axis, then the gradient is zero.

$f'(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x_{1,2} = \pm 1, \quad x = -1 \Rightarrow y = \dfrac{1+1+1}{1-1+1} = 3,\ x = 1 \Rightarrow y = \dfrac{1-1+1}{1+1+1} = \dfrac{1}{3}$

So, the points are: $A(-1, 3)$ and $B\left(1, \dfrac{1}{3}\right)$.

b



On the graph of the first derivative, the stationary points are points where the second derivative is zero, that is, the points of inflexion. A calculator gives us the following:

$x = -1.53$ or $x = -0.347$ or $x = 1.88$

c i We notice that the denominator is never equal to zero, and therefore the domain of the function is the whole set of real numbers. Also, the function has a horizontal asymptote.

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^2 - x + 1 \big/ \div x^2}{x^2 + x + 1 \big/ \div x^2} = \lim_{x \to \pm\infty} \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}} = 1$$

Since the minimum point is below the asymptote and the maximum point is above the asymptote of this continuous function, we can say that the range is $\left[\frac{1}{3}, 3\right]$.

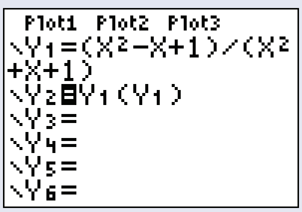
ii $(f \circ f)(x) = f\left(\dfrac{x^2 - x + 1}{x^2 + x + 1}\right) = \dfrac{\left(\dfrac{x^2 - x + 1}{x^2 + x + 1}\right)^2 - \dfrac{x^2 - x + 1}{x^2 + x + 1} + 1}{\left(\dfrac{x^2 - x + 1}{x^2 + x + 1}\right)^2 + \dfrac{x^2 - x + 1}{x^2 + x + 1} + 1}$

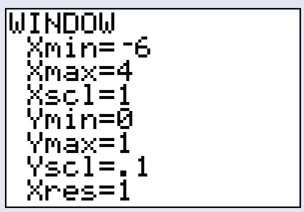
$$= \frac{\dfrac{(x^2 - x + 1)^2 - (x^2 - x + 1)(x^2 + x + 1) + (x^2 + x + 1)^2}{\cancel{(x^2 + x + 1)^2}}}{\dfrac{(x^2 - x + 1)^2 + (x^2 - x + 1)(x^2 + x + 1) + (x^2 + x + 1)^2}{\cancel{(x^2 + x + 1)^2}}}$$

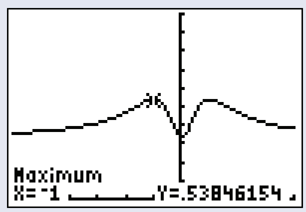
$$= \frac{\cancel{(x^2+1)^2} \cancel{-2x(x^2+1)} + x^2 \cancel{-(x^2+1)^2} + x^2 + (x^2+1)^2 \cancel{+2x(x^2+1)} + x^2}{(x^2+1)^2 \cancel{-2x(x^2+1)} \cancel{+x^2} + (x^2+1)^2 \cancel{-x^2} + (x^2+1)^2 \cancel{+2x(x^2+1)} + x^2} = \frac{(x^2+1)^2 + 3x^2}{3(x^2+1)^2 + x^2}$$

We again notice that the domain is the whole set of real numbers and, since the polynomials in the numerator and denominator are both quartic and the leading coefficients are 1 and 3 respectively, the horizontal asymptote has the equation $y = \frac{1}{3}$.

We can input the composite function into a GDC and find the minimum and maximum point. Notice that, using the GDC, we did not have to find the formula for the composite function; we merely had to define the composition in the graphical mode.







So, the range of the composite function is $\left[\frac{1}{3}, \frac{7}{13}\right]$.

42 $\dfrac{dV}{dt} = 8\text{ cm}^3/\text{s}, V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dr} = 4r^2\pi, \dfrac{dr}{dt} = \dfrac{dr}{dV} \times \dfrac{dV}{dt} = \dfrac{1}{\frac{dV}{dr}} \times \dfrac{dV}{dt} \Rightarrow \dfrac{dr}{dt}(r = 2) = \dfrac{1}{4 \times 2^2\pi} \times 8 = \dfrac{1}{2\pi}$ cm/s

43 $x^3y^2 = 8 \Big/ \dfrac{d}{dx} \Rightarrow 3x^2y^2 + x^3 \times 2yy' = 0 \Rightarrow y' = -\dfrac{3x^2y^{\cancel{2}}}{2x^3\cancel{y}} = -\dfrac{3x^2y}{2x^3} \Rightarrow$

$y'(2, 1) = -\dfrac{3 \times \cancel{2^2} \times 1}{2 \times 2^{\cancel{3}}} = -\dfrac{3}{4}, m_N = -\dfrac{1}{y'(2, 1)} = \dfrac{4}{3} \Rightarrow$ Equation of normal: $y = \dfrac{4}{3}(x - 2) + 1 \Rightarrow y = \dfrac{4}{3}x - \dfrac{5}{3}$

44 a We need to rewrite the expression in product form for ease of calculation of the first and second derivative.

i $f(x) = \dfrac{x^2}{2^x} = x^2 \times 2^{-x} \Rightarrow f'(x) = 2x \times 2^{-x} + x^2 \times 2^{-x} \ln 2 \times (-1) = \dfrac{2x - x^2 \ln 2}{2^x}$

ii $f'(x) = \left(2x - x^2 \ln 2\right) \times 2^{-x} \Rightarrow f''(x) = (2 - 2x\ln 2) \times 2^{-x} + \left(2x - x^2 \ln 2\right) \times 2^{-x} \ln 2 \times (-1) = \dfrac{2 - 4x\ln 2 + x^2 \ln^2 2}{2^x}$

b i $f'(x) = 0 \Rightarrow \dfrac{2x - x^2 \ln 2}{2^x} = 0 \Rightarrow x(2 - x\ln 2) = 0 \Rightarrow x = \dfrac{2}{\ln 2}$

We discard the second solution because of the domain of the function, $x > 0$.

ii $f''\left(\dfrac{2}{\ln 2}\right) = \dfrac{2 - 4\left(\dfrac{2}{\ln 2}\right)\ln 2 + \left(\dfrac{2}{\ln 2}\right)^2 \ln^2 2}{2^{\left(\frac{2}{\ln 2}\right)}} = \dfrac{2 - 8 + 4}{2^{\left(\frac{2}{\ln 2}\right)}} = \dfrac{-2}{2^{\left(\frac{2}{\ln 2}\right)}} < 0$; therefore, we have a maximum value of the function f. We could have tested the nature of the stationary point by using the sign of the first derivative. We notice that the denominator is always positive and that the numerator is a quadratic function with a negative quadratic coefficient; therefore, at $x = \dfrac{2}{\ln 2}$, it changes its sign from positive to negative, which yields the maximum value.

c $f''(x) = 0 \Rightarrow \dfrac{2 - 4x\ln 2 + x^2 \ln^2 2}{2^x} = 0 \Rightarrow 2 - 4x\ln 2 + x^2 \ln^2 2 = 0 \Rightarrow$

$$x_{1,2} = \frac{4\ln 2 \pm \sqrt{16\ln^2 2 - 4 \times \ln^2 2 \times 2}}{2\ln^2 2} = \frac{4\ln 2 \pm \sqrt{8\ln^2 2}}{2\ln^2 2} = \frac{4\ln 2 \pm 2\ln 2\sqrt{2}}{2\ln^2 2} = \frac{2\ln 2\left(2 \pm \sqrt{2}\right)}{2\ln^2 2}$$

$$= \frac{2 \pm \sqrt{2}}{\ln 2} \Rightarrow x = \frac{2 - \sqrt{2}}{\ln 2} \approx 0.845 \text{ or } x = \frac{2 + \sqrt{2}}{\ln 2} \approx 4.93$$

45 a $f(t) = 3\sec^2 t + 5t \Rightarrow f'(t) = 3 \times 2\sec t \times \sec t \times \tan t + 5 = 6\sec^2 t \tan t + 5$

b i $f(\pi) = 3\sec^2 \pi + 5\pi = 3 + 5\pi$ **ii** $f'(\pi) = 6\sec^3 \pi \sin \pi + 5 = 5$

46 a $2xy^2 = x^2 y + 3; x = 1 \Rightarrow 2y^2 = y + 3 \Rightarrow 2y^2 - y - 3 = 0 \Rightarrow (2y - 3)(y + 1) = 0 \Rightarrow$

$\cancel{y = \frac{3}{2}}$ or $y = -1$ because $y < 0$.

b $2xy^2 = x^2 y + 3 \Big/ \dfrac{d}{dx} \Rightarrow 2y^2 + 2x \times 2yy' = 2xy + x^2 y' \Rightarrow y'\left(4xy - x^2\right) = 2xy - 2y^2 \Rightarrow$

$$y' = \frac{2y(x - y)}{x(4y - x)} \Rightarrow y'(1, -1) = \frac{2 \times (-1)(1 + 1)}{1 \times (4 \times (-1) - 1)} = \frac{-4}{-5} = \frac{4}{5}$$

47 a $y = e^{3x}\sin(\pi x) \Rightarrow \dfrac{dy}{dx} = e^{3x} \times 3 \times \sin(\pi x) + e^{3x}\cos(\pi x) \times \pi = e^{3x}(3\sin(\pi x) + \pi\cos(\pi x))$

b $\dfrac{dy}{dx} = 0 \Rightarrow \overbrace{e^{3x}}^{>0}(3\sin(\pi x) + \pi\cos(\pi x)) = 0 \Rightarrow 3\sin(\pi x) + \pi\cos(\pi x) = 0 \Rightarrow \tan(\pi x) = -\dfrac{\pi}{3}$

$\pi x = \arctan\left(-\dfrac{\pi}{3}\right) + k\pi, k \in \mathbb{Z} \Big/ \div \pi \Rightarrow x = \dfrac{1}{\pi}\arctan\left(-\dfrac{\pi}{3}\right) + k, k \in \mathbb{Z}$

Given the condition, for the smallest positive value of x, we take $k = 1$: $x = \dfrac{1}{\pi}\arctan\left(-\dfrac{\pi}{3}\right) + 1 \approx 0.743$

48 $\dfrac{d\theta}{dt} = \dfrac{1}{60}$ rad/s, $\tan\theta = \dfrac{3000}{x} \Rightarrow x = 3000\cot\theta, \dfrac{dx}{d\theta} = -3000\csc^2\theta$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} \Rightarrow \frac{dx}{dt}\left(\theta = \frac{\pi}{3}\right) = -3000\csc^2\left(\frac{\pi}{3}\right) \times \frac{1}{60} = -\overbrace{\cancel{3000}}^{-1000} \times \frac{\cancel{4}}{\cancel{3}} \times \frac{1}{\cancel{60}\,15} = -\frac{1000}{15} = -\frac{200}{3}$$

So, the speed is $\dfrac{200}{3}$ m/s $= 240$ km/hr.

49 a $f(x)=a\left(b+e^{-cx}\right)^{-1}\Rightarrow f'(x)=a\times(-1)\left(b+e^{-cx}\right)^{-2}\times e^{-cx}\times(-c)=ace^{-cx}\left(b+e^{-cx}\right)^{-2}\Rightarrow$

$$f''(x)=ac\left[e^{-cx}\times(-c)\left(b+e^{-cx}\right)^{-2}+e^{-cx}\times(-2)\left(b+e^{-cx}\right)^{-3}\times e^{-cx}\times(-c)\right]$$

$$=\frac{-ac^2e^{-cx}}{\left(b+e^{-cx}\right)^3}\left(b+e^{-cx}-2e^{-cx}\right)=\frac{-ac^2e^{-cx}\left(b-e^{-cx}\right)}{\left(b+e^{-cx}\right)^3}=\frac{ac^2e^{-cx}\left(e^{-cx}-b\right)}{\left(b+e^{-cx}\right)^3}$$

b $f''(x)=0\Rightarrow\dfrac{\overbrace{ac^2e^{-cx}}^{\neq 0}\left(e^{-cx}-b\right)}{\left(b+e^{-cx}\right)^3}=0\Rightarrow e^{-cx}-b=0\Rightarrow e^{-cx}=b\Rightarrow -cx=\ln b\Rightarrow x=-\dfrac{\ln b}{c}$

For $x=-\dfrac{\ln b}{c}\Rightarrow y=\dfrac{a}{b+e^{-c\times\left(-\frac{\ln b}{c}\right)}}=\dfrac{a}{2b}$; so, the point is $\left(-\dfrac{\ln b}{c},\dfrac{a}{2b}\right)$.

c This is a point of inflexion because the second derivative changes its sign at that point. We notice that to the left of zero the expression is positive, whilst to the right the expression is negative.

50 a $2x^2y+3y^2=16: x=1\Rightarrow 2y+3y^2=16\Rightarrow 3y^2+2y-16=0\Rightarrow(3y+8)(y-2)=0$

$\Rightarrow \cancel{y=-\dfrac{8}{3}}$ or $y=2\Rightarrow p=2$, since the condition is $p>0$.

b $2x^2y+3y^2=16\Big/\dfrac{d}{dx}\Rightarrow 4xy+2x^2y'+6yy'=0\Rightarrow y'\left(2x^2+6y\right)=-4xy\Rightarrow$

$$y'=-\frac{\cancel{4}\,2xy}{\cancel{2}\left(x^2+3y\right)}=-\frac{2xy}{x^2+3y}\Rightarrow y'(1,2)=-\frac{2\times1\times2}{1^2+3\times2}=-\frac{4}{7}$$

51 $f(x)=3^x\Rightarrow f'(x)=3^x\ln 3\Rightarrow f''(x)=3^x\ln 3\times\ln 3=3^x\ln^2 3$

$f''(x)=2\Rightarrow 3^x\ln^2 3=2\Rightarrow 3^x=\dfrac{2}{\ln^2 3}\Big/\ln\Rightarrow x\ln 3=\ln\dfrac{2}{\ln^2 3}\Rightarrow$

$$x=\frac{\ln 2-2\ln(\ln 3)}{\ln 3}=\frac{\ln 2-2\ln(\ln 3)}{\ln 3}\approx 0.460$$

52 Let $\measuredangle CAB=\alpha\Rightarrow\tan\alpha=\dfrac{h}{5}\Rightarrow\alpha=\arctan\left(\dfrac{h}{5}\right)\Rightarrow\dfrac{d\alpha}{dh}=\dfrac{5}{25+h^2}$

$\dfrac{dh}{dt}=2$ cm/s, and when the triangle is equilateral: $h=5\sqrt{3}\Rightarrow\dfrac{d\alpha}{dh}\left(5\sqrt{3}\right)=\dfrac{5}{25+75}=\dfrac{1}{20}$

$$\frac{d\alpha}{dt}=\frac{d\alpha}{dh}\times\frac{dh}{dt}\Rightarrow\frac{d\alpha}{dt}\left(h=5\sqrt{3}\right)=\frac{d\alpha}{dh}\times\frac{dh}{dt}=\frac{1}{20}\times 2=\frac{1}{10}\text{ rad/s}$$

53 $y=\ln(2x-1)\Rightarrow\dfrac{dy}{dx}=\dfrac{2}{2x-1}\Rightarrow\dfrac{d^2y}{dx^2}=\dfrac{-4}{(2x-1)^2}$

54 $x^3+y^3-9xy=0\Big/\dfrac{d}{dx}\Rightarrow 3x^2+3y^2y'-9y-9xy'=0\Rightarrow\cancel{3}y'\left(y^2-3x\right)=\cancel{3}\left(3y-x^2\right)\Rightarrow y'=\dfrac{3y-x^2}{y^2-3x}$

$m_N=-\dfrac{1}{y'(2,4)}=-\dfrac{4^2-3\times2}{3\times4-2^2}=-\dfrac{10}{8}=-\dfrac{5}{4}\Rightarrow$ Equation of normal: $y=-\dfrac{5}{4}(x-2)+4\Rightarrow y=-\dfrac{5}{4}x+\dfrac{13}{2}$

55 a $f'(x)=2\sin\left(5x-\dfrac{\pi}{2}\right)\Rightarrow f''(x)=2\cos\left(5x-\dfrac{\pi}{2}\right)\times5=10\cos\left(5x-\dfrac{\pi}{2}\right)$

b This part cannot be solved using the topics that we have covered so far. We need to use a method of integration:

$$f(x)=\int 2\sin\left(5x-\frac{\pi}{2}\right)dx=2\times\left(-\frac{1}{5}\right)\cos\left(5x-\frac{\pi}{2}\right)+c=-\frac{2}{5}\cos\left(5x-\frac{\pi}{2}\right)+c$$

$$f\left(\frac{\pi}{2}\right)=1\Rightarrow-\frac{2}{5}\cos\left(\frac{5\pi}{2}-\frac{\pi}{2}\right)+c=1\Rightarrow-\frac{2}{5}+c=1\Rightarrow c=\frac{7}{5}$$

$$f(x)=-\frac{2}{5}\cos\left(5x-\frac{\pi}{2}\right)+\frac{7}{5}$$

56 $3x^2y + 2xy^2 = 2 \Big/ \dfrac{d}{dx} \Rightarrow 6xy + 3x^2y' + 2y^2 + 2x \times 2yy' = 0 \Rightarrow y'(3x^2 + 4xy) = -(6xy + 2y^2) \Rightarrow y' = -\dfrac{2y^2 + 6xy}{3x^2 + 4xy}$

$m_N = -\dfrac{1}{y'(1,-2)} = \dfrac{3 \times 1^2 + 4 \times 1 \times (-2)}{2 \times (-2)^2 + 6 \times 1 \times (-2)} = \dfrac{-5}{-4} = \dfrac{5}{4}$

57 $f(x) = \dfrac{x^5 + 2}{x}, x \neq 0 \Rightarrow f'(x) = \dfrac{5x^4 \times x - (x^5 + 2) \times 1}{x^2} = \dfrac{4x^5 - 2}{x^2} = (4x^5 - 2)x^{-2}$

$f''(x) = 20x^4 \times x^{-2} + (4x^5 - 2) \times (-2)x^{-3} = \dfrac{20x^5 - 8x^5 + 4}{x^3} = \dfrac{4(3x^5 + 1)}{x^3}$

Point of inflexion: $f''(x) = 0 \Rightarrow \dfrac{4(3x^5 + 1)}{x^3} = 0 \Rightarrow 3x^5 + 1 = 0 \Rightarrow x = -\sqrt[5]{\dfrac{1}{3}}, y = \dfrac{-\frac{1}{3} + 2}{-\sqrt[5]{\frac{1}{3}}} = -\dfrac{5\sqrt[5]{3}}{3}$

Thus, the coordinates are: $\left(-\dfrac{1}{\sqrt[5]{3}}, -\dfrac{5\sqrt[5]{3}}{3}\right) \approx (-0.803, -2.08)$.

58 **a** $n(t) = 650e^{kt}$

Since the number of bacteria double every 20 minutes, there are 1300 bacteria after 20 minutes.

$n(20) = 650e^{k \times 20} = 1300 \Rightarrow e^{20k} = 2 \Rightarrow 20k = \ln 2 \Rightarrow k = \dfrac{\ln 2}{20}$

b $n(t) = 650e^{\frac{\ln 2}{20}t} \Rightarrow \dfrac{dn}{dt} = 650e^{\frac{\ln 2}{20}t} \times \dfrac{\ln 2}{20} = \dfrac{65 \ln 2}{2} e^{\frac{\ln 2}{20}t}$

$\dfrac{dn}{dt}(t = 90) = \dfrac{65}{2} \times (\ln 2)\, e^{\frac{\ln 2}{20} \times 90} = \dfrac{65 \ln 2 \times 16\sqrt{2}}{2} = 520\sqrt{2} \ln 2 \approx 510$ bacteria/min

59 $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c \Rightarrow f''(x) = 6ax + 2b$

Now, we are going to use the conditions:

$f(0) = 2 \Rightarrow d = 2, f'(0) = -3 \Rightarrow c = -3, f(1) = f'(1) \Rightarrow a + b - 3 + 2 = 3a + 2b - 3, f''(-1) = 6 \Rightarrow -6a + 2b = 6$

So, we have to solve the simultaneous equations in a and b.

$$\dfrac{\left.\begin{cases} 2a + b = 2 \\ -3a + b = 3 \end{cases}\right) -}{5a = -1} \Rightarrow a = -\dfrac{1}{5}, b = 2 + \dfrac{2}{5} = \dfrac{12}{5}$$; so, the polynomial is: $f(x) = -\dfrac{1}{5}x^3 + \dfrac{12}{5}x^2 - 3x + 2$.

60 $f(x) = \cos^3(4x + 1), 0 \leqslant x \leqslant 1$

a $f'(x) = 3\cos^2(4x + 1) \times (-\sin(4x + 1)) \times 4 = -12\cos^2(4x + 1)\sin(4x + 1)$

b $f'(x) = 0 \Rightarrow -12\cos^2(4x + 1)\sin(4x + 1) = 0 \Rightarrow \cos(4x + 1) = 0$ or $\sin(4x + 1) = 0$

$\cos(4x + 1) = 0 \Rightarrow 4x + 1 = \dfrac{\pi}{2} + k\pi \Rightarrow x = \dfrac{\pi - 2}{8} + \dfrac{k\pi}{4}$

We notice that two values of x satisfy the domain of the function: $k = 0 \Rightarrow x = \dfrac{\pi - 2}{8}$ and $k = 1 \Rightarrow x = \dfrac{3\pi - 2}{8}$

$\sin(4x + 1) = 0 \Rightarrow 4x + 1 = k\pi \Rightarrow x = \dfrac{-1 + k\pi}{4}$

Only one value of x satisfies the domain of the function: $k = 1 \Rightarrow x = \dfrac{\pi - 1}{4}$

61 $3^{x+y} = x^3 + 3y \Big/ \dfrac{d}{dx} \Rightarrow 3^{x+y} \ln 3 \times (1 + y') = 3x^2 + 3y' \Rightarrow y'(3^{x+y} \ln 3 - 3) = 3x^2 - 3^{x+y} \ln 3 \Rightarrow$

$y' = (3^{x+y} \ln 3 - 3) = \dfrac{3x^2 - 3^{x+y} \ln 3}{3^{x+y} \ln 3 - 3} = \dfrac{x^2 - 3^{x+y-1} \ln 3}{3^{x+y-1} \ln 3 - 1}$

62 $f(x) = \ln(3x + 1), x > -\dfrac{1}{3}$

a $f'(x) = \dfrac{3}{3x + 1}$

b $x = 2 \Rightarrow y = \ln 7; m_N = -\dfrac{1}{f'(2)} = -\dfrac{7}{3} \Rightarrow$ Equation of normal: $y = -\dfrac{7}{3}(x - 2) + \ln 7 \Rightarrow y = -\dfrac{7}{3}x + \dfrac{14}{3} + \ln 7$

63 $y = x \arcsin x, x \in]-1, 1[\Rightarrow \dfrac{dy}{dx} = \arcsin x + \dfrac{x}{\sqrt{1-x^2}} = \arcsin x + x\left(1-x^2\right)^{-\frac{1}{2}}$

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} + x\left(-\frac{1}{\cancel{2}}\right)\left(1-x^2\right)^{-\frac{3}{2}} \times \left(-\cancel{2}x\right) = \frac{2\left(1-x^2\right)+x^2}{\left(1-x^2\right)^{\frac{3}{2}}} = \frac{2-x^2}{\left(1-x^2\right)^{\frac{3}{2}}}$$

64 $e^{xy} - y^2 \ln x = e \Big/ \dfrac{d}{dx} \Rightarrow e^{xy}\left(y + xy'\right) - 2yy'\ln x - y^2 \times \dfrac{1}{x} = 0 \Rightarrow ye^{xy} - \dfrac{y^2}{x} = y'\left(2y\ln x - xe^{xy}\right) \Rightarrow$

$$y' = \frac{xye^{xy} - y^2}{x\left(2y\ln x - xe^{xy}\right)} \Rightarrow y'(1, 1) = \frac{e-1}{-e} = \frac{1-e}{e}$$

65 $f(x) = \dfrac{2x}{x^2+6}, x \geqslant b, b \in \mathbb{R}$

a $f'(x) = \dfrac{2\left(x^2+6\right) - 2x \times 2x}{\left(x^2+6\right)^2} = \dfrac{2x^2+12-4x^2}{\left(x^2+6\right)^2} = \dfrac{12-2x^2}{\left(x^2+6\right)^2}$

b This function needs to be restricted to the interval where every value occurs only once, that is, from the maximum point until the horizontal asymptote, which is the x-axis.

$f'(x) = 0 \Rightarrow \dfrac{12-2x^2}{\left(x^2+6\right)^2} = 0 \Rightarrow 12 - 2x^2 = 0 \Rightarrow x = \sqrt{6}$

We can justify that for $x = \sqrt{6}$ the function has a maximum since the sign of the first derivative changes from positive to negative.

66 $x^2 + xy + y^2 = 3 \Big/ \dfrac{d}{dx} \Rightarrow 2x + y + xy' + 2yy' = 0 \Rightarrow y'(x+2y) = -(2x+y) \Rightarrow y' = -\dfrac{2x+y}{x+2y}$

a $y'(-1, k) = -\dfrac{2\times(-1)+k}{-1+2k} = \dfrac{2-k}{2k-1}$

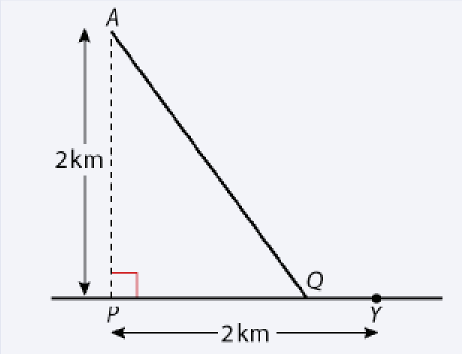
b If the tangent is parallel to the x-axis, then the slope is equal to zero; therefore:

$y'(-1, k) = 0 \Rightarrow \dfrac{2-k}{2k-1} = 0 \Rightarrow 2 - k = 0 \Rightarrow k = 2$

67 $x^3y^2 = \cos(\pi y) \Big/ \dfrac{d}{dx} \Rightarrow 3x^2y^2 + x^3 2yy' = -\sin(\pi y) \times \pi \times y' \Rightarrow y'\left(2x^3y + \pi\sin(\pi y)\right) = -3x^2y^2 \Rightarrow$

$$y' = -\frac{3x^2y^2}{2x^3y + \pi\sin(\pi y)} \Rightarrow y'(-1, 1) = -\frac{-3}{2 + \pi\underbrace{\sin(\pi)}_{0}} = \frac{3}{2}$$

68



a $AQ^2 = 4 + x^2 \Rightarrow T = T_S + T_R = 5\sqrt{5} \times \sqrt{4+x^2} + 5 \times (2-x) = 5\sqrt{5}\sqrt{4+x^2} + 10 - 5x$ minutes

b $\dfrac{dT}{dx} = 5\sqrt{5}\dfrac{\cancel{2}x}{\cancel{2}\sqrt{4+x^2}} - 5 = \dfrac{5\sqrt{5}\,x}{\sqrt{4+x^2}} - 5$

c **i** $\frac{dT}{dx} = 0 \Rightarrow \frac{5\sqrt{5}\,x}{\sqrt{4+x^2}} - 5 = 0 \Rightarrow \left(5\sqrt{5}\,x = 5\sqrt{4+x^2}\right) / \div 5 \Rightarrow 5x^2 = 4 + x^2 \Rightarrow 4x^2 = 4 \Rightarrow x = 1$

We have only one solution since the distance must be positive.

ii $x = 1 \Rightarrow T = 5\sqrt{5}\sqrt{4+1} + 10 - 5 = 30$ minutes

iii In the case of a complicated rational expression, it is simpler to use the product rule.

$$\frac{dT}{dx} = 5\sqrt{5}\,x\left(4+x^2\right)^{-\frac{1}{2}} - 5 \Rightarrow \frac{d^2T}{dx^2} = 5\sqrt{5}\left(4+x^2\right)^{-\frac{1}{2}} + 5\sqrt{5}\,x \times \left(-\frac{1}{\cancel{2}}\right)\left(4+x^2\right)^{-\frac{3}{2}} \times \cancel{2}x$$

$$= \frac{5\sqrt{5}}{\left(4+x^2\right)^{\frac{3}{2}}}\left(4 \cancel{+x^2} \cancel{-x^2}\right) = \frac{20\sqrt{5}}{\left(4+x^2\right)^{\frac{3}{2}}}$$

$\frac{d^2T}{dx^2}(1) = \frac{20\sqrt{5}}{(4+1)^{\frac{3}{2}}} = \frac{20\sqrt{5}}{5\sqrt{5}} = 4 > 0$; therefore, the time found is a minimum.

69 $f(x) = xe^{2x} \Rightarrow f^{(n)}(x) = \left(2^n x + n \times 2^{n-1}\right)e^{2x}, n \in \mathbb{Z}^+$

a $f'(x) = (2x+1)\underbrace{e^{2x}}_{>0} = 0 \Rightarrow 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$

$f''(x) = (4x+4)e^{2x} \Rightarrow f''\left(-\frac{1}{2}\right) = \left(4 \times \left(-\frac{1}{2}\right) + 4\right)e^{2\times\left(-\frac{1}{2}\right)} = 2e^{-1} > 0$; therefore, it is a minimum.

b $f''(x) = (4x+4)\underbrace{e^{2x}}_{>0} = 0 \Rightarrow 4x + 4 = 0 \Rightarrow x = -1$

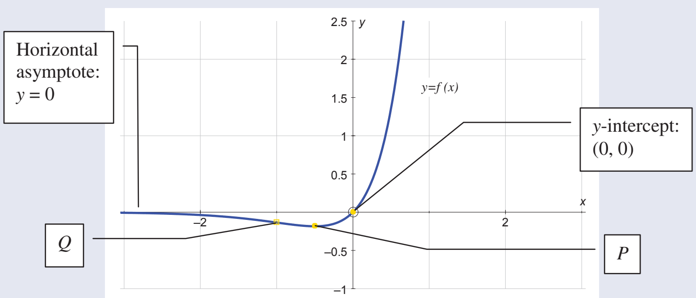
c Since the sign of the second derivative depends on the expression $4x + 4$:

i We can see that the function is concave up when: $f''(x) > 0 \Rightarrow x > -1$.

ii The function is concave down when: $f''(x) < 0 \Rightarrow x < -1$.

d To be able to clearly sketch $y = f(x)$, we need to find the y-coordinates of points P and Q.

$f\left(-\frac{1}{2}\right) = -\frac{1}{2} \times e^{-1} \approx -0.184, f(-1) = -1 \times e^{-2} \approx -0.135$



e **Basis step:** $n = 1 \Rightarrow f'(x) = (2x+1)e^{2x}$

Alternatively, we can differentiate the function and check the result:

$f(x) = xe^{2x} \Rightarrow f'(x) = e^{2x} + xe^{2x} \times 2 = e^{2x}(1 + 2x)$

We can conclude that the formula works for $n = 1$.

Inductive step: We assume that the formula works for $n = k \Rightarrow f^{(k)}(x) = \left(2^k x + k \times 2^{k-1}\right)e^{2x}$

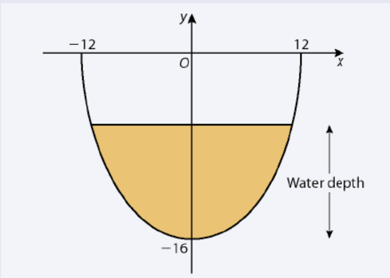
We need to see whether the formula works for:

$$n = k+1 \Rightarrow f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = \left(\left(2^k x + k \times 2^{k-1}\right)e^{2x}\right)' = 2^k e^{2x} + \left(2^k x + k \times 2^{k-1}\right)e^{2x} \times 2$$

$$= e^{2x}\left(2^k + 2^{k+1}x + k \times 2^k\right) = e^{2x}\left(2^{k+1}x + (1+k) \times 2^k\right) = e^{2x}\left(2^{k+1}x + (k+1) \times 2^{(k+1)-1}\right)$$

This is the formula for $n = k + 1$.

The formula works for $n = 1$ and, from the assumption that it works for $n = k$, we determined that it works for $n = k + 1$. Therefore, by the principle of mathematical induction, we conclude that it works for all $n \in \mathbb{Z}^+$.

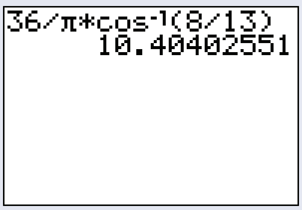
70 

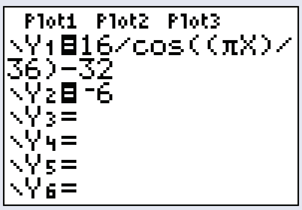
In order to find the width, we need to find the intersection between the curve and the horizontal line $y = -6$, since the water depth is 10 m.

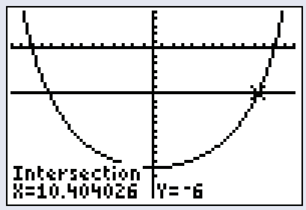
$$16 \sec\left(\frac{\pi x}{36}\right) - 32 = -6 \Rightarrow \sec\left(\frac{\pi x}{36}\right) = \frac{13}{8} \Rightarrow \cos\left(\frac{\pi x}{36}\right) = \frac{8}{13} \Rightarrow x = \frac{36}{\pi} \arccos\left(\frac{8}{13}\right)$$

The width is twice as long, so: $w = \dfrac{72}{\pi} \arccos\left(\dfrac{8}{13}\right)$ cm.

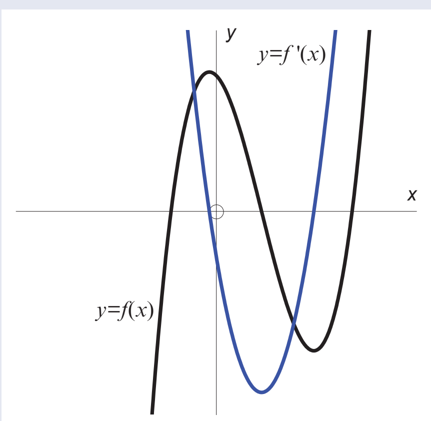
In cases such as this, if we have enough time, it is advisable to check the answer by using a GDC.

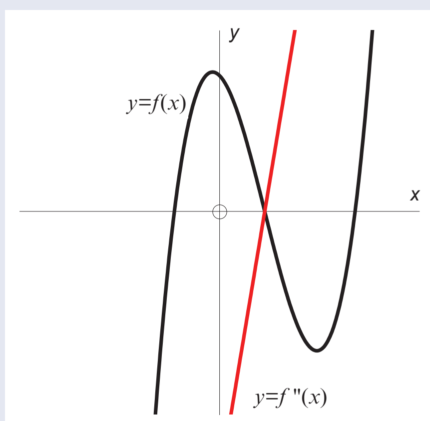






71 By looking at the graph of the function, we can conclude that the function is a cubic polynomial; therefore, its first derivative will be a quadratic polynomial with zeros at the x-coordinates of the stationary points and the minimum at the x-coordinate of the point of inflexion. The reason for stating the minimum point of the first derivative is the positive cubic coefficient of the original function. The second derivative will be a straight line with a positive slope and the zero at the x-coordinate of the point of inflexion.

a) 

b) 

Chapter 16

Practice questions

1 **a** We know that the parameter p is directly related to the amplitude; therefore, we can find that $p = 3$.

b $\int_0^{\frac{\pi}{2}} 3\cos x \, dx = [3\sin x]_0^{\frac{\pi}{2}} = 3\sin\left(\frac{\pi}{2}\right) - 3\sin 0 = 3$

Note: Even though you might not know how to find the parameter in part **a**, it is always advisable to proceed with part **b** and attempt to write the definite integral.

2 **a** $y = e^{\frac{x}{2}} \Rightarrow y(0) = e^{\frac{0}{2}} = 1$; therefore, point P has the coordinates (0, 1).

b $V = \pi\int_0^{\ln 2}\left(e^{\frac{x}{2}}\right)^2 dx = \pi\int_0^{\ln 2} e^x \, dx$

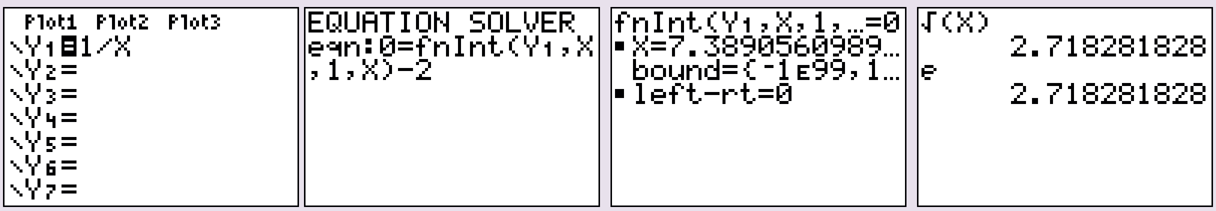
c $\pi\int_0^{\ln 2} e^x \, dx = \pi\left[e^x\right]_0^{\ln 2} = \pi\left(e^{\ln 2} - e^0\right) = \pi(2-1) = \pi$

Solution Paper 1 type

3 $\int_1^a \frac{1}{x} dx = 2 \Rightarrow [\ln|x|]_1^a = 2 \Rightarrow \ln a - \ln 1 = 2 \Rightarrow \ln a = 2 \Rightarrow a = e^2$

Solution Paper 2 type

3



Notice that we changed variable a into variable x, and that Solver accepts all of the features from the calculator's menu with only one variable parameter to be represented as a function in that variable. We could have used the graphical mode, but, since there is no more than one solution, we were satisfied using Solver. At the end, we can check whether the numerical result is a special value which we could have recognized.

4 **a** $y = \ln x \Rightarrow y' = \frac{1}{x}$. At the point $(e, 1)$ the slope of the tangent is: $m = y'(e) = \frac{1}{e}$. The tangent can be found by using the formula for the tangent: $y = f'(x_1)(x - x_1) + y_1$, where (x_1, y_1) is a particular point on the graph of the function.

$y = \frac{1}{e}(x - e) + 1 \Rightarrow y = \frac{1}{e}x \cancel{-1} \cancel{+1} \Rightarrow y = \frac{1}{e}x$

Since the linear function has no y-intercept, it means that it passes through the origin. If we input the coordinates of the origin into the equation, we get a **true statement**: $0 = \frac{1}{e} \times 0 \Rightarrow 0 = 0$.

b For the first term, we need to apply the product rule.

$(x\ln x - x)' = \ln x + \cancel{x} \times \frac{1}{\cancel{x}} - 1 = \ln x + 1 - 1 = \ln x$

c The shaded region can be split into two. The first region is a triangle bounded by the tangent line, x-axis and the vertical line $x = 1$. Since that is a right-angled triangle, the area is calculated as: $A_{\text{Triangle}} = \frac{1 \cdot \frac{1}{e}}{2} = \frac{1}{2e}$.

In order to find the area of the second region, we need to evaluate the following integral:

$$\int_1^e\left(\frac{1}{e}x-\ln x\right)dx=\left[\frac{1}{e}\times\frac{x^2}{2}-(x\ln x-x)\right]_1^e=\left(\frac{1}{\cancel{e}}\times\frac{e^{\cancel{2}}}{2}-e\overbrace{\ln e}^{1}+e\right)-\left(\frac{1}{e}\times\frac{1^2}{2}-1\overbrace{\ln 1}^{0}+1\right)=\frac{1}{2}e-\frac{1}{2e}-1$$

Now, the total area is the sum of those two areas; therefore, $A=\frac{1}{2e}+\frac{1}{2}e-\frac{1}{2e}-1=\frac{1}{2}e-1$.

Solution Paper 1 type

5 a i $s(t)=800+100t-4t^2 \Rightarrow s(5)=800+100\times5-4\times5^2=1200$ m

ii $v(t)=s'(t)\Rightarrow v(t)=100-8t\Rightarrow v(5)=100-8\times5=60$ m s^{-1}

iii $v(t)=36\Rightarrow 100-8t=36\Rightarrow 64=8t\Rightarrow t=8$ s

iv $s(8)=800+100\times8-4\times8^2=1344$ m

b Firstly, we need to find the time at which the plane stops after touchdown:

$v(t)=0\Rightarrow 100-8t=0\Rightarrow t=\frac{\cancel{100}\,25}{\cancel{8}\,2}=12.5$ s

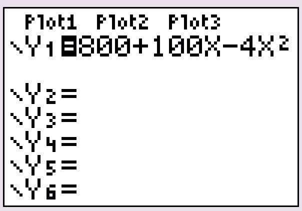
Now, we need to find the distance the plane will travel after touchdown:

$s\left(\frac{25}{2}\right)-s(0)=\cancel{800}+\cancel{100}\,50\times\frac{25}{\cancel{2}}-\cancel{4}\times\left(\frac{25}{\cancel{2}}\right)^2\cancel{-800}=1250-625=625$ m

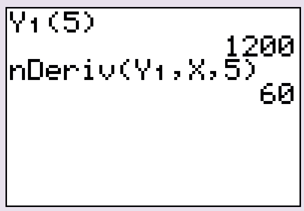
Using part **a iv**, the remaining runway length is $2000-1344=656$ m; therefore, there is enough runway to stop the plane if it makes a touchdown before point P.

Solution Paper 2 type

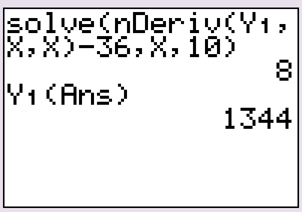
5 a



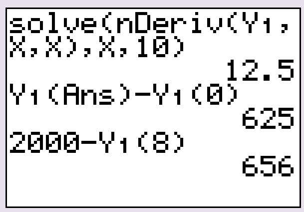
i and **ii**



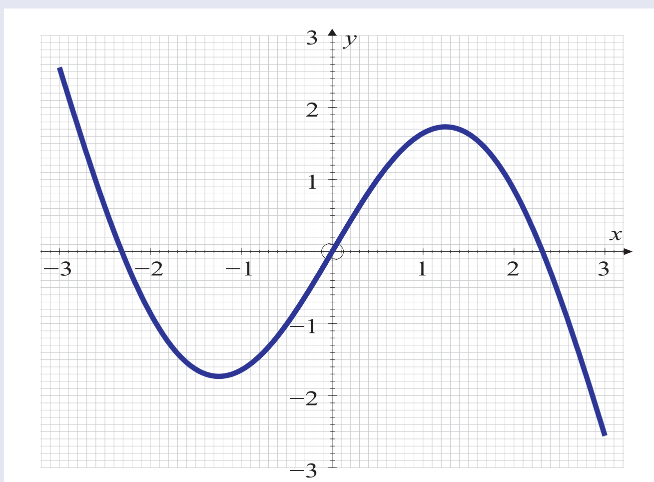
iii and **iv**

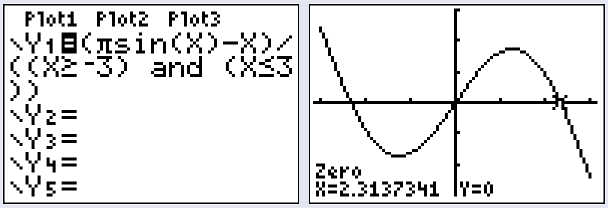


b



6 a To draw the function, we input the function into the calculator and then use Table to plot the points (or, alternatively, use the trace feature).



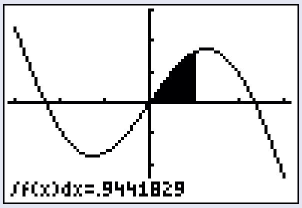
b 

Note: Parts **a** and **b** cannot be solved without using a calculator.

c $\int(\pi\sin x - x)\,dx = -\pi\cos x - \dfrac{x^2}{2} + c, c \in \mathbb{R}$

$$\text{Area} = \int_0^1 (\pi\sin x - x)\,dx = \left[-\pi\cos x - \frac{x^2}{2}\right]_0^1 = \left(-\pi\cos 1 - \frac{1}{2}\right) - \left(-\pi\overbrace{\cos 0}^{1} - 0\right)$$

$$= \pi(1-\cos 1) - \frac{1}{2} \approx 0.944, \text{ correct to three significant figures.}$$

Otherwise: 

7 Method I: From the direction of the x-axis

We split the shaded region into two: R_1, the rectangle enclosed by the lines $x = 0$, $y = \frac{4}{3}$, $x = 1$ and $y = 2$; and R_2, the region enclosed by $x = 1$, $y = \frac{4}{3}$ and $y = 1 + \frac{1}{x}$. The area of the first region is: $A_1 = \left(2 - \frac{4}{3}\right) \times (1-0) = \frac{2}{3}$. To find the area of the second region, we need to do two things: firstly, we have to find the point of intersection of the line $y = \frac{4}{3}$ and the curve $y = 1 + \frac{1}{x}$. By inspection, we see that the point of intersection is $\left(3, \frac{4}{3}\right)$. Now, to be able to use the definite integral, we need to translate the graph vertically $\frac{4}{3}$ units down.

$$A_2 = \int_1^3 \left(1 + \frac{1}{x} - \frac{4}{3}\right) dx = \int_1^3 \left(\frac{1}{x} - \frac{1}{3}\right) dx = \left[\ln|x| - \frac{1}{3}x\right]_1^3$$

$$= (\ln 3 - 1) - \left(\overbrace{\ln 1}^{0} - \frac{1}{3}\right) = \ln 3 - \frac{2}{3} \Rightarrow A = A_1 + A_2 = \frac{2}{3} + \ln 3 - \frac{2}{3} = \ln 3$$

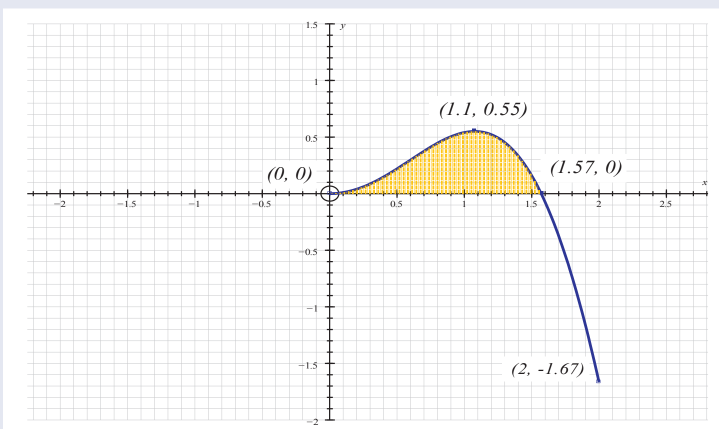
Method II: From the direction of the y-axis

We need to express x in terms of y and calculate the integral with respect to the y-variable.

$$y = 1 + \frac{1}{x} \Rightarrow \frac{1}{x} = y - 1 \Rightarrow x = \frac{1}{y-1}$$

$$A = \int_{\frac{4}{3}}^{2} \frac{1}{y-1}\,dy = \int_{\frac{4}{3}}^{2} \frac{1}{y-1}\,d(y-1) = \left[\ln|y-1|\right]_{\frac{4}{3}}^{2} = \ln 1 - \ln\frac{1}{3} = 0 - (-\ln 3) = \ln 3$$

8 a **i** and **ii** To draw the function, we input the function into the calculator and then use Table to plot the points (or, alternatively, use the trace feature).



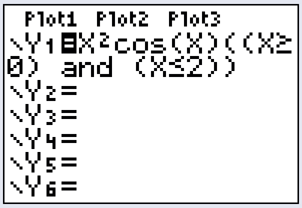
b $x^2 \cos x = 0, x > 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$

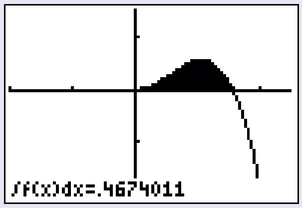
c **i** See the diagram in part **a**. **ii** $\int_0^{\frac{\pi}{2}} x^2 \cos x \, dx$

d $\int_0^{\frac{\pi}{2}} x^2 \cos x \, dx = \left[x^2 \sin x + 2x \cos x - 2 \sin x\right]_0^{\frac{\pi}{2}}$

$$= \left(\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)\overbrace{\cos\left(\frac{\pi}{2}\right)}^{0} - 2\sin\left(\frac{\pi}{2}\right)\right) - \left(0^2 \sin(0) + 2 \times 0 \times \cos(0) - 2\sin(0)\right) = \frac{\pi^2}{4} - 2 \approx 0.467$$

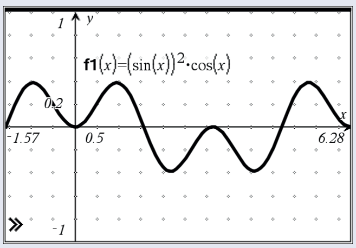
Or:





9 **a** $f(x + 2\pi) = \sin^2(x + 2\pi)\cos(x + 2\pi) = \sin^2(x)\cos(x) = f(x)$, so the fundamental period of f is 2π.

b



By looking at the graph, we estimate that the range would be $[-0.4, 0.4]$.

c **i** $f'(x) = (2\sin x \cos x)\cos x + \sin^2 x(-\sin x) = 2\sin x \cos^2 x - \sin^3 x$. We can also continue to express the whole derivative in terms of sine only:

$2\sin x \cos^2 x - \sin^3 x = 2\sin x\left(1 - \sin^2 x\right) - \sin^3 x = 2\sin x - 3\sin^3 x$

ii In this problem, we use the first form of the derivative:

$f'(x) = 0 \Rightarrow 2\sin x \cos^2 x - \sin^3 x = 0 \Rightarrow \sin x\left(2\cos^2 x - \sin^2 x\right) = 0$

Since the value of sine cannot be equal to 0 at A, we can conclude that:

$2\cos^2 x - \sin^2 x = 0 \Rightarrow 2\cos^2 x - \left(1 - \cos^2 x\right) = 0 \Rightarrow 3\cos^2 x - 1 = 0 \Rightarrow \cos x = \sqrt{\frac{1}{3}}$

iii $f(x_{max}) = \sin^2(x_{max})\cos(x_{max}) = \left(1 - \frac{1}{3}\right) \times \sqrt{\frac{1}{3}} = \frac{2}{3\sqrt{3}}$ or $\frac{2\sqrt{3}}{9}$

d $f(x) = 0 \Rightarrow \sin x = 0$ or $\cos x = 0 \Rightarrow x = 0, x = \pi$ or $x = \frac{\pi}{2}$, so the x-coordinate of point B is $\frac{\pi}{2}$.

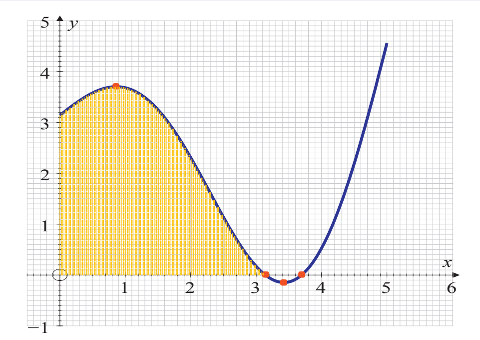
e **i** $\int \sin^2(x)\cos(x)\,dx = \int (\sin x)^2 d(\sin x) = \frac{\sin^3 x}{3} + c, c \in \mathbb{R}$

ii $\int_0^{\frac{\pi}{2}} f(x)\,dx = \left[\frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{2}} = \frac{\sin^3\left(\frac{\pi}{2}\right)}{3} - \frac{\sin^3 0}{3} = \frac{1}{3}$

f $f''(x) = 0 \Rightarrow 9\cos^3 x - 7\cos x = 0 \Rightarrow \cos x\left(9\cos^2 x - 7\right) = 0$. Since the x-coordinate of C is less than $\frac{\pi}{2}$, the second factor must be equal to 0.

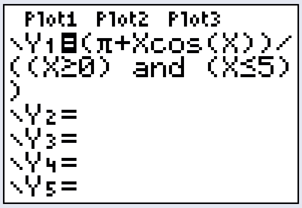
$9\cos^2 x - 7 = 0 \Rightarrow \cos^2 x = \frac{7}{9} \Rightarrow \cos x = \frac{\sqrt{7}}{3} \Rightarrow x = \arccos\left(\frac{\sqrt{7}}{3}\right) \approx 0.491$

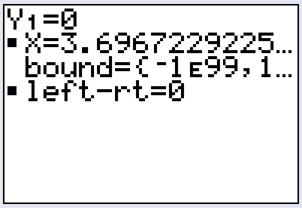
10 a



b $x = \pi \Rightarrow \pi + \pi \underbrace{\cos \pi}_{-1} = \pi - \pi = 0$

c





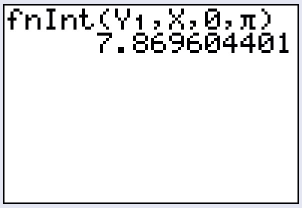
In this problem, if we use Solver, we need to use an estimated value that is further to the right of π, which we found as the first zero. Our estimate was 4. So, the answer is 3.696 72.

d See the diagram above.

Area $= \int_0^{\pi} (\pi + x\cos x)\,dx$

e Area $= \int_0^{\pi} (\pi + x\cos x)\,dx = [\pi x + x\sin x + \cos x]_0^{\pi} = (\pi^2 + 0 - 1) - (0 + 0 + 1)$

$= \pi^2 - 2 \approx 7.869\,044\,01$. Or, by using a GDC, we get:



So, the answer, correct to six significant figures, is 7.869 60.

11 a i $p = g(x) - f(x) = (10x + 2) - (1 + e^{2x}) = 10x + 1 - e^{2x}$

ii $p' = 10 - 2e^{2x} = 0 \Rightarrow 2e^{2x} = 10 \Rightarrow e^{2x} = 5 \Rightarrow 2x = \ln 5 \Rightarrow x = \frac{\ln 5}{2} \approx 0.805$ (3 s.f.)

b i $x = 1 + e^{2y} \Rightarrow e^{2y} = x - 1 \Rightarrow 2y = \ln(x-1) \Rightarrow y = \frac{1}{2}\ln(x-1) = \ln\sqrt{x-1}$

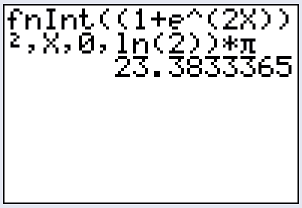
ii $f^{-1}(x) = \ln\sqrt{x-1} \Rightarrow f^{-1}(5) = \ln\sqrt{5-1} = \ln 2$

c $V = \pi\int_0^{\ln 2} (1 + e^{2x})^2\,dx$

Bonus: Evaluate the integral in part **c**:

$V = \pi\int_0^{\ln 2} (1 + e^{2x})^2\,dx = \pi\int_0^{\ln 2} (1 + 2e^{2x} + e^{4x})\,dx = \pi\left[x + e^{2x} + \frac{e^{4x}}{4}\right]_0^{\ln 2}$

$= \pi\left[\left(\ln 2 + e^{2\ln 2} + \frac{e^{4\ln 2}}{4}\right) - \left(0 + e^0 + \frac{e^0}{4}\right)\right] = \pi\left(\ln 2 + 4 + \frac{\cancel{16}\,4}{\cancel{4}} - 1 - \frac{1}{4}\right) = \pi\left(\ln 2 + \frac{27}{4}\right) \approx 23.4$



12 It is not possible to solve this question with a GDC.

$$V = \pi\int_0^a \left((ax+2)^2 - \left(x^2+2\right)^2\right)dx = \pi\int_0^a \left(a^2x^2 + 4ax + 4 - x^4 - 4x^2 - 4\right)dx$$

$$= \pi\int_0^a \left(\left(a^2-4\right)x^2 + 4ax - x^4\right)dx = \pi\left(\frac{\left(a^2-4\right)x^3}{3} + 2ax^2 - \frac{x^5}{5}\right)\Bigg]_0^a = \pi\left(\frac{a^5-4a^3}{3} + 2a^3 - \frac{a^5}{5}\right) = \pi\left(\frac{2a^5}{15} + \frac{2a^3}{3}\right)$$

13 $$\int x\sqrt{\frac{1}{2}x+1}\,dx = \left[\begin{array}{c} u = \frac{1}{2}x + 1 \Rightarrow x = 2u - 2 \\ du = \frac{1}{2}dx \Rightarrow dx = 2\,du \end{array}\right] = \int (2u-2)\sqrt{u}\times 2\,du = 4\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= 4\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + c = \frac{8}{5}\left(\frac{1}{2}x+1\right)^{\frac{5}{2}} - \frac{8}{3}\left(\frac{1}{2}x+1\right)^{\frac{3}{2}} + c,\ c\in\mathbb{R}$$

14 $$a = \frac{dv}{ds}\times\frac{ds}{dt} = \frac{dv}{ds}\times v \Rightarrow a = \frac{3(2s-1)-2(3s+2)}{(2s-1)^2}\times\frac{3s+2}{2s-1} = \frac{-7(3s+2)}{(2s-1)^3}$$

$$s = 2 \Rightarrow a = \frac{-7(6+2)}{(4-1)^3} = -\frac{56}{27}$$

15 $$V = \pi\int_0^k \left(e^x\right)^2 dx = \pi\int_0^k e^{2x}\,dx = \pi\times\left(\frac{1}{2}e^{2x}\right)\Bigg]_0^k = \frac{\pi\left(e^{2k}-1\right)}{2}$$

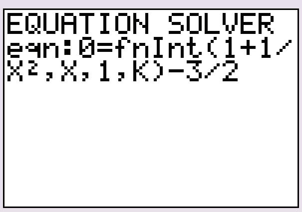
Solution Paper 1 type

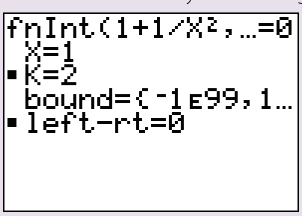
16 $$\int_1^k\left(1+\frac{1}{x^2}\right)dx = \left(x - \frac{1}{x}\right)\Bigg]_1^k = k - \frac{1}{k}\cancel{-1}\cancel{+1} = \frac{k^2-1}{k} = \frac{3}{2} \Rightarrow 2k^2 - 3k - 2 = 0 \Rightarrow (2k+1)(k-2) = 0$$

$\cancel{k = -\frac{1}{2}}$ or $k = 2$, since $k > 1$.

Solution Paper 2 type

16 We will use Solver on a GDC. We could have easily used the graphing menu too, in a very similar manner.





Note: The value of x, shown in the final screen, is not relevant and it can be any value.

17 We can deduce that $a(t) = -\frac{1}{20}t + 2, v(0) = 0$. So, we can now proceed with finding the distance travelled by the train.

$$v(t) = \int\left(-\frac{1}{20}t + 2\right)dt = -\frac{1}{40}t^2 + 2t + c,\ c\in\mathbb{R}$$

$$v(0) = 0 \Rightarrow c = 0 \Rightarrow v(t) = -\frac{1}{40}t^2 + 2t$$

$$d = \int_0^{60}\left(-\frac{1}{40}t^2 + 2t\right) = \left(-\frac{t^3}{120} + t^2\right)\Bigg]_0^{60} = -1800 + 3600 = 1800 \text{ m}$$

18 Firstly, we need to find the zeros of the parabola.

$y = a^2 - x^2 \Rightarrow y = (a - x)(a + x) \Rightarrow x_1 = -a, x_2 = a$

The area of the rectangle is $A_R = 2ah$, where h is the height of the rectangle.

The area under the parabola is calculated by the following integral.

$$A_P = \int_{-a}^{a} (a^2 - x^2)\,dx = \left(a^2 x - \frac{x^3}{3}\right)\Bigg]_{-a}^{a} = \left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right) = \frac{4}{3}a^3$$

Since the two areas must be equal, we can find the height of the rectangle:

$$2ah = \frac{4}{3}a^3 \Rightarrow h = \frac{2}{3}a^2$$

So, the dimensions of the rectangle are: $2a \times \frac{2}{3}a^2$.

19 a $f_k(x) = x \ln x - kx, x > 0 \Rightarrow f_k'(x) = \ln x + 1 - k, x > 0$

b If the function is increasing, the first derivative is positive; therefore:

$\ln x + 1 - k > 0 \Rightarrow \ln x > k - 1 \Rightarrow x > e^{k-1}, x \in \left]e^{k-1}, +\infty\right[$

The question asks us to find the interval over which $f(x)$ is increasing; therefore, the value of k is 0 and the interval is:

$x > e^{-1} = \frac{1}{e}, x \in \left]\frac{1}{e}, +\infty\right[$.

c i $f_k'(x) = \ln x + 1 - k = 0 \Rightarrow \ln x = k - 1 \Rightarrow x = e^{k-1}$

ii $f_k(x) = x \ln x - kx = 0 \Rightarrow x(\ln x - k) = 0 \Rightarrow x = 0$ or $\ln x - k = 0$

So, the other x-intercept is: $\ln x - k = 0 \Rightarrow \ln x = k \Rightarrow x = e^k$

d Since the curve is below the x-axis, we need to take the absolute value of the integral.

$\int_0^{e^k} (x \ln x - kx)\,dx$

To solve the first part of the integral, we need to use integration by parts.

$$\int x \ln x\,dx = \begin{bmatrix} u = \ln x & du = \frac{1}{x}dx \\ dv = x\,dx & v = \frac{x^2}{2} \end{bmatrix} = \frac{x^2}{2}\ln x - \int\left(\frac{x^{\cancel{2}}}{2} \times \frac{1}{\cancel{x}}\right)dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

$$\int_0^{e^k} (x \ln x - kx)\,dx = \left(\frac{x^2}{2}\ln x - \frac{x^2}{4} - k\frac{x^2}{2}\right)\Bigg]_0^{e^k} = \frac{e^{2k}}{4}\left(2\ln e^k - 1 - 2k\right) = -\frac{e^{2k}}{4}$$

So, the area enclosed by the curve and the x-axis is: $\frac{e^{2k}}{4}$.

e $A\left(e^k, 0\right)$, $m = f_k'\left(e^k\right) = \underbrace{\ln e^k}_{k} + 1 - k = 1 \Rightarrow$ Equation of tangent: $y = 1 \times \left(x - e^k\right) + 0 \Rightarrow y = x - e^k$

f The y-intercept is $-e^k$, so the area of the triangle enclosed by the tangent and the coordinate axes is:

$A = \frac{1}{2}\left|e^k \times \left(-e^k\right)\right| = \frac{e^{2k}}{2} = 2 \times \frac{e^{2k}}{4}$, which is twice the area enclosed by the curve.

g $k = 1 \Rightarrow x_1 = e, k = 2 \Rightarrow x_1 = e^2, k = 3 \Rightarrow x_1 = e^3, k = 4 \Rightarrow x_1 = e^4, \ldots$

To verify the statement, we are going to take two consecutive x-intercepts, for k and $k + 1$:

$\frac{x_{k+1}}{x_k} = \frac{e^{k+1}}{e^k} = e$. The ratio is constant and therefore the zeros form a geometric sequence.

20 $\dfrac{dy}{dx} = 1 + y^2 \Rightarrow \dfrac{dy}{1+y^2} = dx \Rightarrow \displaystyle\int \dfrac{dy}{1+y^2} = \int dx \Rightarrow \arctan y = x + c$

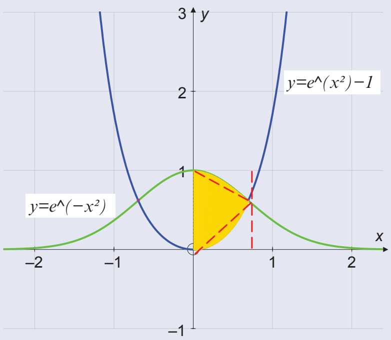
$y = 0, x = 2 \Rightarrow \arctan 0 = 2 + c \Rightarrow c = -2 \Rightarrow \arctan y = x - 2 \Rightarrow y = \tan(x-2)$

21 $kx = mv\dfrac{dv}{dx} \Rightarrow kx\,dx = mv\,dv \Rightarrow \displaystyle\int kx\,dx = \int mv\,dv \Rightarrow k\dfrac{x^2}{2} = m\dfrac{v^2}{2} + c, c \in \mathbb{R}$

$x = 0, v = v_0 \Rightarrow k \times 0 = m\dfrac{v_0^2}{2} + c \Rightarrow c = -\dfrac{mv_0^2}{2} \Rightarrow k\dfrac{x^2}{2} = m\dfrac{v^2}{2} - \dfrac{mv_0^2}{2} \Rightarrow k\dfrac{x^2}{2} = \dfrac{m\left(v^2 - v_0^2\right)}{2}$

$x = 2 \Rightarrow k\dfrac{2^2}{2} = \dfrac{m\left(v^2 - v_0^2\right)}{2} \Rightarrow \dfrac{4k}{m} = v^2 - v_0^2 \Rightarrow v = \sqrt{\dfrac{4k}{m} + v_0^2}$

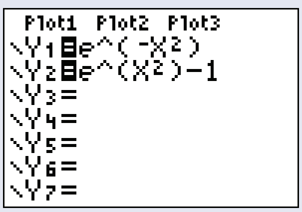
22 a

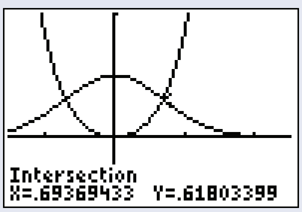


b Given that the point of intersection has an x-coordinate equal to p, we notice that the rectangle has dimensions $1 \times p$ and the triangle has a vertical base of length 1 and height p.

$A_{TRIANGLE} < A_{REGION} < A_{RECTANGLE} \Rightarrow \dfrac{1}{2}p \times 1 < A_{REGION} < p \times 1 \Rightarrow \dfrac{p}{2} < A_{REGION} < p$

c

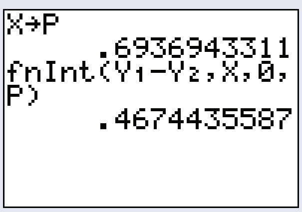




$p = 0.6937$ (correct to four decimal places).

d $A_{REGION} = \int_0^p \left(e^{-x^2} - \left(e^{x^2} - 1\right)\right)dx$

In a case like this, it is advisable to store the coordinates of the point of intersection in the GDC's memory and then work with this more accurate value.



So, the area of the region is 0.467 (correct to three significant figures).

Notice that the last two parts of the question cannot be done without using a calculator.

23 a $\displaystyle\int x\cos 3x\,dx = \begin{bmatrix} u = x & du = dx \\ dv = \cos 3x\,dx & v = \frac{1}{3}\sin 3x \end{bmatrix} = \frac{1}{3}x\sin 3x - \int \frac{1}{3}\sin 3x\,dx$

$= \dfrac{1}{3}x\sin 3x - \dfrac{1}{3} \times \left(-\dfrac{1}{3}\cos 3x\right) + c = \dfrac{1}{3}x\sin 3x + \dfrac{1}{9}\cos 3x + c, c \in \mathbb{R}$

b **i** $\int_{\frac{\pi}{6}}^{\frac{3\pi}{6}} x\cos 3x\,dx = \left(\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right)\Big]_{\frac{\pi}{6}}^{\frac{3\pi}{6}}$

$$= \left(\frac{\pi}{6}\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{-1} + \frac{1}{9}\underbrace{\cos\left(\frac{3\pi}{2}\right)}_{0}\right) - \left(\frac{\pi}{18}\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} + \frac{1}{9}\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}\right) = -\frac{2\pi}{9} \Rightarrow A = \frac{2\pi}{9}$$

The value of the integral is negative, but, as we need to calculate the area, we simply take the absolute value of the integral, since the function is always negative for the given integral.

ii $\int_{\frac{3\pi}{6}}^{\frac{5\pi}{6}} x\cos 3x\,dx = \left(\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right)\Big]_{\frac{3\pi}{6}}^{\frac{5\pi}{6}} = \left(\frac{5\pi}{18}\underbrace{\sin\left(\frac{5\pi}{2}\right)}_{1} + \frac{1}{9}\underbrace{\cos\left(\frac{5\pi}{2}\right)}_{0}\right) - \left(\frac{\pi}{6}\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{-1} + \frac{1}{9}\underbrace{\cos\left(\frac{3\pi}{2}\right)}_{0}\right) = \frac{4\pi}{9}$

iii $\int_{\frac{5\pi}{6}}^{\frac{7\pi}{6}} x\cos 3x\,dx = \left(\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right)\Big]_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}$

$$= \left(\frac{7\pi}{18}\underbrace{\sin\left(\frac{7\pi}{2}\right)}_{-1} + \frac{1}{9}\underbrace{\cos\left(\frac{7\pi}{2}\right)}_{0}\right) - \left(\frac{5\pi}{18}\underbrace{\sin\left(\frac{5\pi}{2}\right)}_{1} + \frac{1}{9}\underbrace{\cos\left(\frac{5\pi}{2}\right)}_{0}\right) = -\frac{6\pi}{9} \Rightarrow A = \frac{6\pi}{9}$$

Again, the integral was negative, so, for the area, we take the absolute value.

c The areas enclosed by the given boundaries form an arithmetic sequence with first term $u_1 = \frac{2\pi}{9}$ and common difference $d = \frac{2\pi}{9}$. Therefore, the sum of the first n terms is given by:

$$S_n = n \times \frac{2\pi}{9} + \frac{n(n-1)}{2} \times \frac{2\pi}{9} = \frac{2n\pi}{9}\left(1 + \frac{n-1}{2}\right) = \frac{2n\pi}{9} \times \frac{1+n}{2} = \frac{n(n+1)\pi}{9}, n \in \mathbb{Z}^+$$

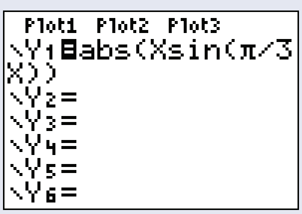
24 **a** $v(t) = 0 \Rightarrow t\sin\left(\frac{\pi}{3}t\right) = 0 \Rightarrow t = 0$ or $\frac{\pi}{3}t = k\pi \Rightarrow t = 3k, k \in \mathbb{Z}$

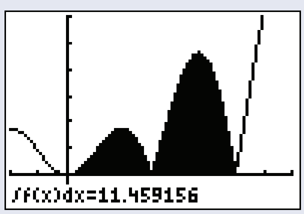
Using the restricted domain, we can calculate the values of t: $t = 0$ or $t = 3$ or $t = 6$.

b **i** In order to avoid a discussion of the positive or negative values of the parts of the integral, we will simply use the absolute values.

Total distance travelled $= \int_0^6 \left| t\sin\left(\frac{\pi}{3}t\right) dt \right|$

ii





So, the total distance travelled is 11.5 m (correct to three significant figures).

Note: If not using a calculator, we should split the integral into two parts, from 0 to 3 and from 3 to 6, where the last one has a negative value and we take its opposite value. The anti-derivative can be found by using integration by parts.

25 **a** Distance travelled $= \int_0^1 v(t)\,dt = \int_0^1 \frac{1}{2+t^2}\,dt = \left(\frac{1}{\sqrt{2}}\arctan\left(\frac{t}{\sqrt{2}}\right)\right)\Big]_0^1$

$$= \left(\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\right)\right) - \left(\frac{1}{\sqrt{2}}\underbrace{\arctan 0}_{0}\right) = \frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\right) \approx 0.435\text{ m}$$

b $a = \dfrac{dv}{dt} \Rightarrow a(t) = \dfrac{-2t}{\left(2+t^2\right)^2}$

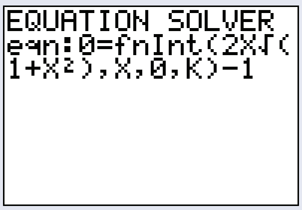
26 a $y = 2x\sqrt{1+x^2} \Rightarrow \dfrac{dy}{dx} = 2\sqrt{1+x^2} + 2x \times \dfrac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}} = 2\sqrt{1+x^2} + \dfrac{2x^2}{\sqrt{1+x^2}}$

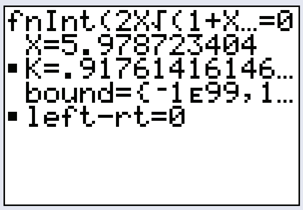
b $\int 2x\sqrt{1+x^2}\,dx = \begin{bmatrix} u = 1+x^2 \\ du = 2x\,dx \end{bmatrix} = \int \sqrt{u}\,du = \dfrac{2}{3}u^{\frac{3}{2}} + c = \dfrac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + c, c \in \mathbb{R}$

c $\int_0^k 2x\sqrt{1+x^2}\,dx = \left(\dfrac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}\right)\Bigg]_0^k = \dfrac{2}{3}\left(1+k^2\right)^{\frac{3}{2}} - \dfrac{2}{3} \Rightarrow \dfrac{2}{3}\left(1+k^2\right)^{\frac{3}{2}} - \dfrac{2}{3} = 1 \Rightarrow \left(1+k^2\right)^{\frac{3}{2}} = \dfrac{5}{2} \Rightarrow$

$1+k^2 = \left(\dfrac{5}{2}\right)^{\frac{2}{3}} \Rightarrow k^2 = \left(\dfrac{5}{2}\right)^{\frac{2}{3}} - 1 \Rightarrow k = \sqrt{\left(\dfrac{5}{2}\right)^{\frac{2}{3}} - 1} \approx 0.918$

This part can be solved directly by using a GDC.





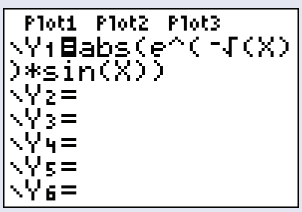
Again, the value of x is irrelevant for this calculation. To be correct, we need to say that the value of x must be within the domain of the function. In our case, the domain is the set of all real numbers.

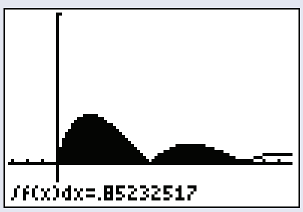
27 $v(t) = 6t^2 - 6t, t \geqslant 0 \Rightarrow \text{distance} = \int_0^2 \left|6t^2 - 6t\right| dt = \int_0^1 \left(6t - 6t^2\right) dt + \int_1^2 \left(6t^2 - 6t\right) dt$

$= \left(3t^2 - 2t^3\right)\Big]_0^1 + \left(2t^3 - 3t^2\right)\Big]_1^2 = (1-0) + (16 - 12 - 2 + 3) = 6 \text{ m}$

The curve is a parabola which opens upwards, with the zeros at 0 and 1; therefore, the function is negative from 0 to 1 and positive from 1 to 2.

28





The total distance travelled is 0.852 m (correct to three significant figures).

29 a $\dfrac{dT}{dt} = k(T-22) \Rightarrow \dfrac{dT}{T-22} = k\,dt \Rightarrow \int \dfrac{dT}{T-22} = \int k\,dt \Rightarrow$

$\ln|T-22| = kt + c \Rightarrow T - 22 = e^{kt+c}, c \in \mathbb{R} \Rightarrow T = 22 + Ae^{kt}, A \in \mathbb{R}^+$

b To find the constants A and k, we need to solve the simultaneous equations formed from the given information.

i $\begin{cases} T(0) = 100 \\ T(15) = 70 \end{cases} \Rightarrow \begin{cases} 22 + Ae^0 = 100 \\ 22 + Ae^{15k} = 70 \end{cases} \Rightarrow \begin{cases} A = 78 \\ 78e^{15k} = 48 \end{cases} \Rightarrow \begin{cases} A = 78 \\ e^{15k} = \dfrac{8}{13} \end{cases} \Rightarrow \begin{cases} A = 78 \\ 15k = \ln\left(\dfrac{8}{13}\right) \end{cases} \Rightarrow$

$\begin{cases} A = 78 \\ k = \dfrac{1}{15}\ln\left(\dfrac{8}{13}\right) \approx -0.324 \end{cases}$

ii $T = 22 + 78e^{\frac{\ln\left(\frac{8}{13}\right)t}{15}} \Rightarrow 40 = 22 + 78e^{\frac{\ln\left(\frac{8}{13}\right)t}{15}} \Rightarrow \frac{18}{78} = e^{\frac{\ln\left(\frac{8}{13}\right)t}{15}} \Rightarrow \ln\left(\frac{3}{13}\right) = \frac{\ln\left(\frac{8}{13}\right)t}{15} \Rightarrow$

$$\ln\left(\frac{3}{13}\right) = \frac{\ln\left(\frac{8}{13}\right)t}{15} \Rightarrow t = \frac{15\ln\left(\frac{3}{13}\right)}{\ln\left(\frac{8}{13}\right)} \approx 45.3$$

30 $x\frac{dy}{dx} - y^2 = 1 \Rightarrow x\frac{dy}{dx} = y^2 + 1 \Rightarrow \frac{dy}{y^2+1} = \frac{dx}{x} \Rightarrow \arctan(y) = \ln|x| + c,\ c \in \mathbb{R}$

$$y = 0,\ x = 2 \Rightarrow \underbrace{\arctan(0)}_{0} = \ln 2 + c \Rightarrow c = -\ln 2$$

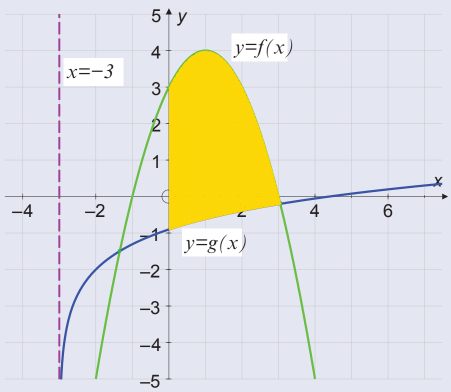
$$\arctan(y) = \ln|x| - \ln 2 \Rightarrow \arctan(y) = \ln\left|\frac{x}{2}\right|$$

$$y = \tan\left(\ln\left|\frac{x}{2}\right|\right)$$

31 $\int \frac{x^3}{(x+2)^2}\,dx = \left[\begin{array}{c} u = x+2 \\ du = dx \end{array}\right] = \int \frac{(u-2)^3}{u^2}\,du = \int \frac{u^3 - 6u^2 + 12u - 8}{u^2}\,du$

$$= \int\left(u - 6 + \frac{12}{u} - \frac{8}{u^2}\right)du = \frac{u^2}{2} - 6u + 12\ln|u| + \frac{8}{u} + c = \frac{(x+2)^2}{2} - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + c,\ c \in \mathbb{R}$$

32 a

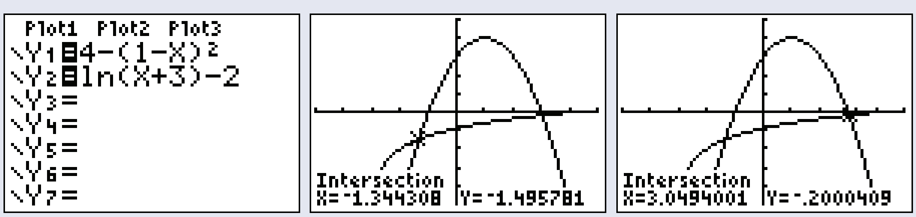


b **i** The logarithmic function $g(x) = \ln(x+3) - 2$ has a vertical asymptote: $x = -3$.

ii y-intercept: $x = 0 \Rightarrow g(x) = \ln(3) - 2 \approx -0.901$

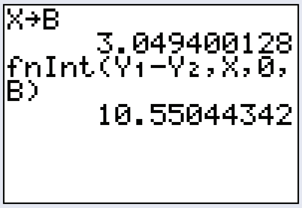
x-intercept: $y = 0 \Rightarrow 0 = \ln(x+3) - 2 \Rightarrow \ln(x+3) = 2 \Rightarrow x + 3 = e^2 \Rightarrow x = e^2 - 3 \approx 4.39$

c



d **i** Refer to the diagram in part **a**.

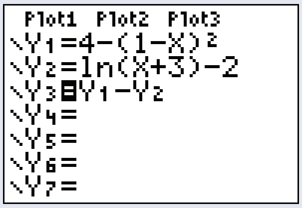
ii $\int_0^{3.05}\left(\left(4 - (1-x)^2\right) - \left(\ln(x+3) - 2\right)\right)dx$

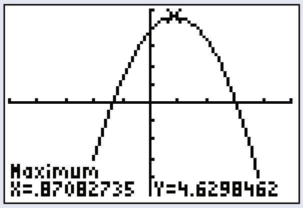
iii 

So, the shaded region has an area of 10.6 (correct to three significant figures).

Note: We were able to store the x-coordinate of the intersection in the GDC's memory since that was the last calculation done before we found the integral.

e To find the maximum distance, we need to form a new function: $h(x) = f(x) - g(x)$.





So, the maximum distance between $f(x)$ and $g(x)$ is 4.63 (correct to three significant figures).

33 a $x = e^{\theta} \Rightarrow dx = e^{\theta} d\theta \Rightarrow d\theta = \dfrac{dx}{x}, \dfrac{dy}{d\theta} = \dfrac{y}{e^{2\theta}+1} \Rightarrow \dfrac{x\,dy}{dx} = \dfrac{y}{x^2+1} \Rightarrow \dfrac{dy}{y} = \dfrac{dx}{x(x^2+1)} \Rightarrow \displaystyle\int \dfrac{dy}{y} = \int \dfrac{dx}{x(x^2+1)}$

b $\displaystyle\int \frac{dx}{x(x^2+1)} = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \ln|y| - \frac{1}{2}\ln(x^2+1) + c, c \in \mathbb{R}$

c $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x(x^2+1)} \Rightarrow \ln|y| = \ln|x| - \frac{1}{2}\ln(x^2+1) + c \Rightarrow \ln|y| = \ln|e^{\theta}| - \frac{1}{2}\ln(e^{2\theta}+1) + c, c \in \mathbb{R}$

$\theta = 0, y = \sqrt{2} \Rightarrow \dfrac{1}{2}\ln 2 = \underbrace{\ln 1}_{0} - \dfrac{1}{2}\ln 2 + c \Rightarrow c = \ln 2$

$\ln|y| = \ln|e^{\theta}| - \dfrac{1}{2}\ln(e^{2\theta}+1) + \ln 2 \Rightarrow y = \dfrac{2e^{\theta}}{\sqrt{e^{2\theta}+1}}$

Chapter 17

Practice questions

1 Let X be the amount of savings (\$), where $X \sim \mathrm{N}\left(\mu = 3000, \sigma^2 = 500^2\right)$.

a $\mathrm{P}(X > 3200) = \mathrm{P}\left(Z > \dfrac{3200 - 3000}{500}\right) = 1 - \mathrm{P}(Z < 0.4) = 1 - 0.6554 = 0.3446 = 34.5\%$

b $\mathrm{P}(2300 < \mathrm{X} < 3300) = \mathrm{P}\left(\dfrac{2300 - 3000}{500} < Z < \dfrac{3300 - 3000}{500}\right) = \mathrm{P}(-1.4 < Z < 0.6)$

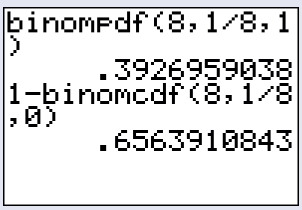
$$= \mathrm{P}(Z < 0.6) - (1 - \mathrm{P}(Z < 1.4)) = 0.7257 - (1 - 0.9192) = 0.6449$$

So, for two townspeople, we need to square the result: $0.6449^2 \approx 0.416$.

c $\mathrm{P}(X < d) = 0.7422 \Rightarrow \mathrm{P}\left(Z < \dfrac{d - 3000}{500}\right) = 0.7422 \Rightarrow \dfrac{d - 3000}{500} = 0.6495 \Rightarrow d = 3325$

2 **a** Let X be the number of black discs, where $X \sim \mathrm{B}\left(n = 8, p = \dfrac{5}{35+5} = \dfrac{1}{8}\right)$.

a i–ii



b Now, we change the number of trials to 400. Hence, the expected number of black discs that would be drawn is $= 400 \cdot \dfrac{1}{8} = 50$.

3 **a** The area of the shaded region is 0.1.

b Since the areas are the same, $\mathrm{P}(X \geqslant 12) = 0.1$ and $\mathrm{P}(X \leqslant 8) = 0.1$. We can find the mean value as the average of the two numbers: $= \dfrac{12+8}{2} = 10$.

c To find the standard deviation we need to use the tables.

$\mathrm{P}(X \leqslant 8) = 0.1 \Rightarrow \mathrm{P}\left(Z \leqslant -\dfrac{2}{\sigma}\right) = 0.1 \Rightarrow \mathrm{P}\left(Z \leqslant \dfrac{2}{\sigma}\right) = 0.9 \Rightarrow \dfrac{2}{\sigma} = 1.2816 \Rightarrow \sigma = 1.56$

d $\mathrm{P}(X \leqslant 11) = \mathrm{P}\left(Z \leqslant \dfrac{11-10}{1.56}\right) = \mathrm{P}(Z \leqslant 0.641) = 0.739$ (correct to 3 s.f.)

4 Let X be the number of heads obtained, where $\mathrm{X} \sim \mathrm{B}\left(n = 8, p = \dfrac{1}{2}\right)$.

a $\mathrm{P}(X = 4) = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \times \left(\dfrac{1}{2}\right)^4 \times \left(\dfrac{1}{2}\right)^4 = \dfrac{\cancel{8} \times 7 \times \cancel{6} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \times \dfrac{1}{2^{\cancel{8}\,7}} = \dfrac{35}{128}$

b $\mathrm{P}(X = 3) = \begin{pmatrix} 8 \\ 3 \end{pmatrix} \times \left(\dfrac{1}{2}\right)^3 \times \left(\dfrac{1}{2}\right)^5 = \dfrac{\cancel{8} \times 7 \times \cancel{6}}{1 \times \cancel{2} \times \cancel{3}} \times \dfrac{1}{2^{\cancel{8}\,5}} = \dfrac{7}{32}$

c Since this binomial expression is symmetrical with respect to the middle observation of obtaining 4 heads, the probability of obtaining 3 heads and the probability of obtaining 5 heads are equal.

$\mathrm{P}(3 \leqslant X \leqslant 5) = 2 \cdot \dfrac{7}{32} + \dfrac{35}{128} = \dfrac{56 + 35}{128} = \dfrac{91}{128}$

5 Let X be the lifespan of an insect, where $X \sim \mathrm{N}\left(\mu = 57, \sigma^2 = 4.4^2\right)$.

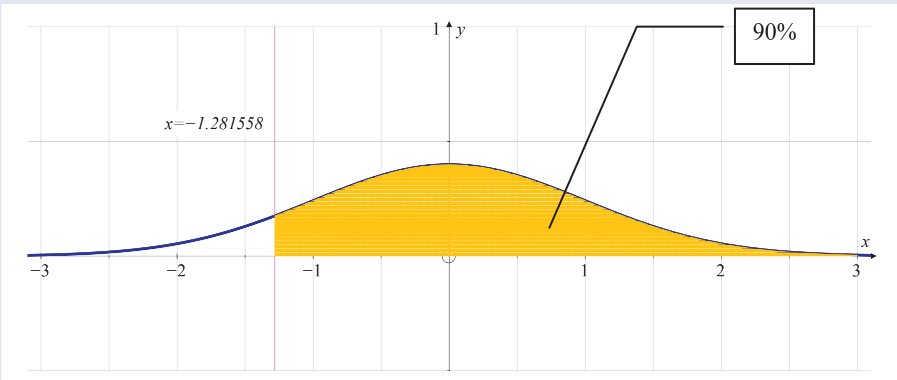
a We need to calculate the values of a and b by using the transformation to standard normal variable.

$$a = \frac{55 - 57}{4.4} \approx -0.455,\ b = \frac{60 - 57}{4.4} \approx 0.682$$

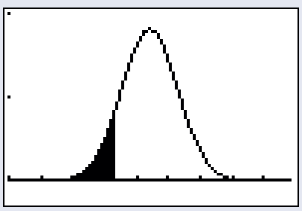
b i–ii

```
normalcdf(55,101
,57,4.4)
         .6752818401
normalcdf(55,60,
57,4.4)
         .4276049496
```

c i If 90% of the insects die after t hours, that means that 10% of the insects have a lifespan of up to t hours.



ii



```
EQUATION SOLVER
eqn:0=normalcdf(
13,X,57,4.4)-.1
```

```
normalcdf(13,…=0
 X=51.361164271…
 bound={-1E99,1…
```

6 a $\mathrm{P}(X > 50) = 0.3 \Rightarrow \mathrm{P}\left(Z > \frac{50-\mu}{10}\right) = 0.3 \Rightarrow \mathrm{P}\left(Z < \frac{50-\mu}{10}\right) = 0.7$

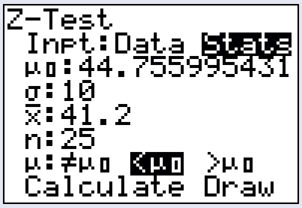
```
EQUATION SOLVER
eqn:0=normalcdf(
-1000,50,M,10)-.
7
```

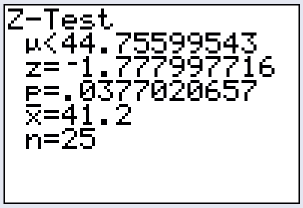
```
normalcdf(-10…=0
▪M=44.755995431…
 bound={-1E99,1…
▪left-rt=0
```

b H_1: The mean speed has been affected by the campaign.

c A one-tailed test is appropriate since the police are interested in decreasing the speed.

d To solve this part of the problem, we are going to use the Z-Test.

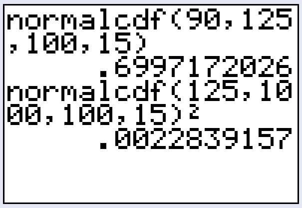




Since $p = 0.0377 < 0.05$, we reject H_0.

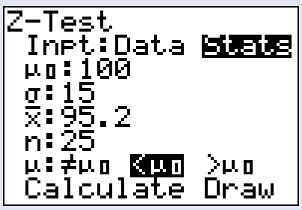
7 Let X be a person's IQ, where $X \sim \mathrm{N}\left(\mu = 100, \sigma^2 = 15^2\right)$.

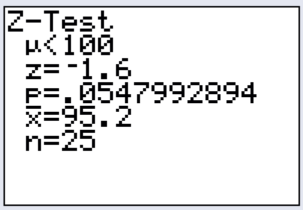
a–b



c H_0: The average IQ of the group suffering from the disorder is 100.

H_1: The average IQ of the group suffering from the disorder is less than 100.

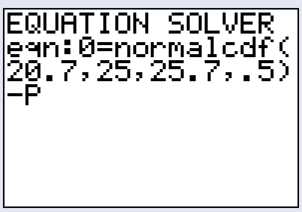


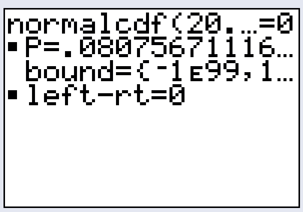


Since $p = 0.0548 > 0.05$, we do not have enough evidence to reject the null hypothesis H_0.

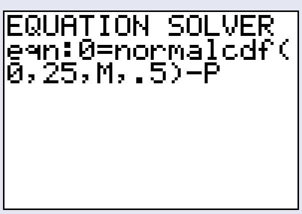
8 Let X be the weight of a bag, where $X \sim \mathrm{N}\left(\mu = 25.7, \sigma^2 = 0.5^2\right)$.

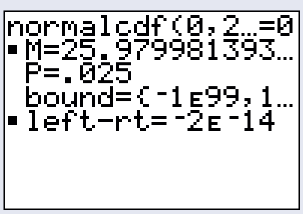
a



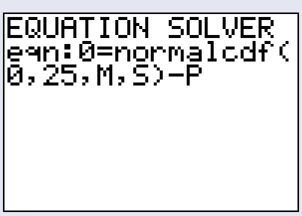


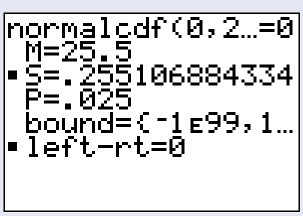
b





c Since the probability of a bag weighing less than 25 kg is the same as the probability of a bag weighing more than 26 kg, the mean value must be = 25.5 kg.

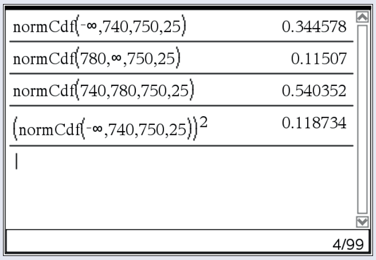




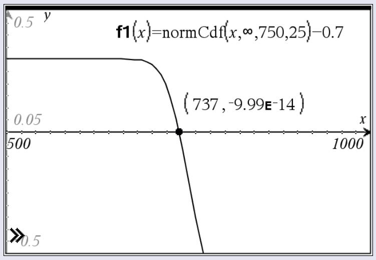
d With a mean of 26 kg, we lose 0.5 kg per bag; therefore, we lose $0.5 \cdot 0.8 = 0.4$ dollars per bag. For an initial cost of \$5000, we need to sell $\frac{5000}{0.4} = 12\,500$ bags to cover the investment.

9 Let X be the mass of the packets, where $X \sim \mathrm{N}\left(\mu = 750, \sigma^2 = 25^2\right)$.

a–b

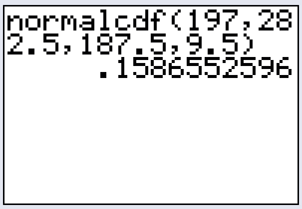


c

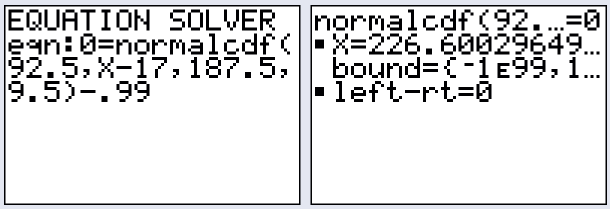


10 Let X be the height of adults in Tallopia, where $X \sim \mathrm{N}\left(\mu = 187.5, \sigma^2 = 9.5^2\right)$.

a



b



11 Let X be the mass of a lion, where $X \sim \mathrm{N}\left(\mu = 310, \sigma^2 = 30^2\right)$.

a $\mathrm{P}(X \geqslant 350) = \mathrm{P}\left(Z \geqslant \frac{350 - 310}{30}\right) = 1 - \mathrm{P}\left(Z < \frac{4}{3}\right) = 1 - 0.9082 = 0.0918$

b Since a and b are symmetrical with respect to the mean, we can write:

$$\mathrm{P}(a \leqslant X \leqslant b) = \mathrm{P}\left(\frac{a - 310}{30} \leqslant Z \leqslant \frac{b - 310}{30}\right) = 2 \times \mathrm{P}\left(Z < \frac{b - 310}{30}\right) - 1 = 0.95 \Rightarrow$$

$$\mathrm{P}\left(Z < \frac{b - 310}{30}\right) = 0.975 \Rightarrow \frac{b - 310}{30} = 1.96 \Rightarrow b = 368.8 \Rightarrow a = 251.2$$

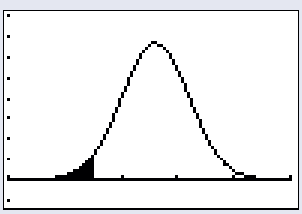
12 Let X be the reaction time measured in seconds, where $X \sim \mathrm{N}\left(\mu = 0.76, \sigma^2 = 0.06^2\right)$.

a $a = \frac{0.7 - 0.76}{0.06} = -1$ and $b = \frac{0.79 - 0.76}{0.06} = 0.5$

b **i** $\mathrm{P}(X > 0.7) = \mathrm{P}(Z > -1) = \mathrm{P}(Z < 1) = 0.8413$

ii $\mathrm{P}(0.7 < X < 0.79) = \mathrm{P}(-1 < Z < 0.5) = \mathrm{P}(Z < 0.5) - (1 - \mathrm{P}(Z < -1)) = 0.6915 - 0.1587 = 0.5328$

c **i**



ii $\mathrm{P}(X < c) = \mathrm{P}\left(Z < \frac{c - 0.76}{0.06}\right) = 0.03 \Rightarrow \mathrm{P}\left(Z < \frac{0.76 - c}{0.06}\right) = 0.97 \Rightarrow \frac{0.76 - c}{0.06} = 1.8808 \Rightarrow$

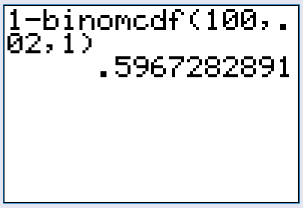
$c = 0.76 - 1.8808 \times 0.06 = 0.647\,152$

13 Let X be the number of faulty calculators, where $X \sim \mathrm{B}(n = 100, p = 0.02)$.

a $\mathrm{E}(X) = np \Rightarrow \mathrm{E}(X) = 100 \times 0.02 = 2$

b $\mathrm{P}(X=3)=\binom{100}{3}0.02^3\times 0.98^{97}=0.182\,276$

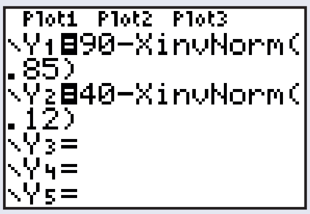
c Using the complementary event: $\mathrm{P}(X\geqslant 2)=1-\mathrm{P}(X\leqslant 1)$

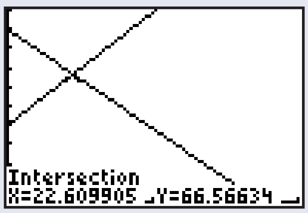


14 We need to set up a system of two equations with two unknowns, but this time we will use a slightly different approach.

$$\left.\begin{aligned}\mathrm{P}(X>90)=0.15\\ \mathrm{P}(X<40)=0.12\end{aligned}\right\}\Rightarrow\left.\begin{aligned}\mathrm{P}\left(Z>\frac{90-\mu}{\sigma}\right)=0.15\\ \mathrm{P}\left(Z<\frac{40-\mu}{\sigma}\right)=0.12\end{aligned}\right\}\Rightarrow\left.\begin{aligned}\mathrm{P}\left(Z<\frac{90-\mu}{\sigma}\right)=0.85\\ \mathrm{P}\left(Z<\frac{40-\mu}{\sigma}\right)=0.12\end{aligned}\right\}\Rightarrow$$

$$\left.\begin{aligned}\frac{90-\mu}{\sigma}=\mathrm{invNorm}(0.85)\\ \frac{40-\mu}{\sigma}=\mathrm{invNorm}(0.12)\end{aligned}\right\}\Rightarrow\left.\begin{aligned}90-\sigma\mathrm{invNorm}(0.85)=\mu\\ 40-\sigma\mathrm{invNorm}(0.012)=\mu\end{aligned}\right\}$$

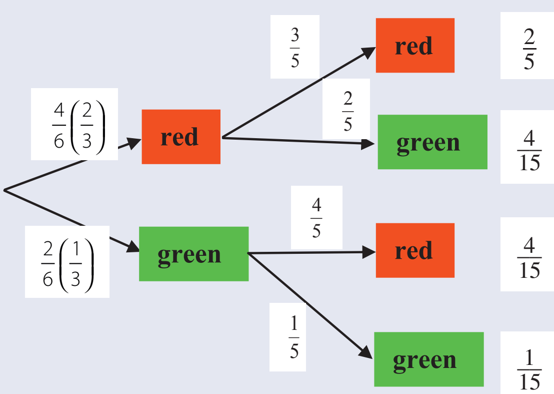




The x- and y-values are σ and μ respectively.

15 a $\mathrm{E}(X)=\sum_i x_i p_i\Rightarrow \mu=0\times\frac{3}{10}+1\times\frac{6}{10}+\cancel{2}\times\frac{1}{\cancel{10}\,5}=\frac{8}{10}=\frac{4}{5}$

b i



ii

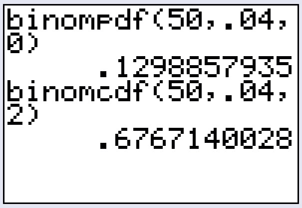
y	0	1	2
$\mathrm{P}(Y=y)$	$\frac{1}{15}$	$\frac{8}{15}$	$\frac{2}{5}$

c $\mathrm{P}(2\mathrm{R})=\underbrace{\frac{2}{6}}_{1\text{ or }6}\cdot\underbrace{\frac{1}{10}}_{\text{Bag A}}+\underbrace{\frac{4}{6}}_{2,3,4,5}\cdot\underbrace{\frac{2}{5}}_{\text{Bag B}}=\frac{9}{30}=\frac{3}{10}$

d $\mathrm{P}(\mathrm{A}\,|\,2\mathrm{R})=\dfrac{\frac{\cancel{2}\,1}{\cancel{6}\,3}\cdot\frac{1}{\cancel{10}}}{\frac{3}{\cancel{10}}}=\frac{1}{9}$

16 Let X be the number of defective ball bearings, where $X \sim \mathrm{B}(n = 50, p = 0.04)$.

a–b

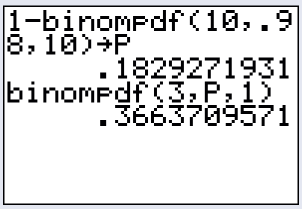


c $\mathrm{E}(X) = np \Rightarrow \mathrm{E}(X) = 50 \times 0.04 = 2$

17 Let X be the number of functioning CDs in a pack, where $X \sim \mathrm{B}(n = 10, p = 0.98)$.

a Using the complementary event, the probability that a package is returned is 0.1829 (see GDC screen below).

b Introducing a new binomial variable: $Y \sim \mathrm{B}(n = 3, p = 0.182\,93)$



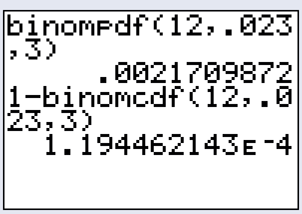
18 a $\sum \mathrm{P}(x) = 1 \Rightarrow 2k + 2k^2 + k^2 + k + 2k^2 + k = 1 \Rightarrow 5k^2 + 4k = 1 \Rightarrow 5k^2 + 4k - 1 = 0 \Rightarrow$

$(5k - 1)(k + 1) = 0 \Rightarrow 5k - 1 = 0 \text{ or } k + 1 = 0 \Rightarrow k = \frac{1}{5} \text{ or } \overbrace{k = -1}^{\text{Impossible}}$

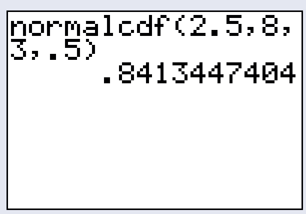
Note: As a probability, k must be a positive number between 0 and 1.

b $\mathrm{E}(X) = \sum_i x_i p_i \Rightarrow \mu = 0 \times \frac{2}{5} + 1 \times \frac{2}{25} + 2 \times \frac{6}{25} + 3 \times \frac{7}{25} = \frac{35}{25} = \frac{7}{5} = 1.4$

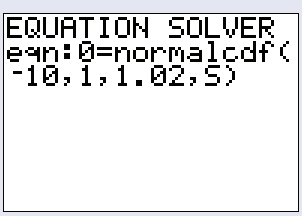
19 a Let X be the number of small tomatoes, where $X \sim \mathrm{B}(n = 12, p = 0.023)$.

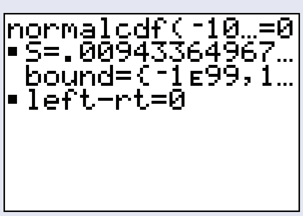


b Let Y be the size of a tomato, where $Y \sim \mathrm{N}(\mu = 3, \sigma^2 = 0.5^2)$.



20 $X \sim \mathrm{N}(\mu = 1.02, \sigma) \Rightarrow \mathrm{P}(X < 1) = 0.017$





Since the units in the calculation are kg, the standard deviation is 9.4 g.

21 $f(x) = \begin{cases} e - ke^{kx} & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$

a $\int_0^1 \left(e - ke^{kx}\right)dx = 1 \Rightarrow \left(ex - e^{kx}\right)\Big]_0^1 = 1 \Rightarrow e - e^k + 1 = 1 \Rightarrow e = e^k \Rightarrow k = 1$

b $\int_{\frac{1}{4}}^{\frac{1}{2}} \left(e - e^x\right)dx = \left(ex - e^x\right)\Big]_{\frac{1}{4}}^{\frac{1}{2}} = \frac{1}{2}e - e^{\frac{1}{2}} - \frac{1}{4}e + e^{\frac{1}{4}} = \frac{1}{4}e - \sqrt{e} + \sqrt[4]{e}$

c To find the two values, we need to refresh our memory regarding integration by parts.

$$\int xe^x\,dx = \begin{bmatrix} u = x & du = dx \\ dv = e^x\,dx & v = e^x \end{bmatrix} = xe^x - \int e^x\,dx = xe^x - e^x + c,\ c \in \mathbb{R}$$

$$\int x^2e^x\,dx = \begin{bmatrix} u = x^2 & du = 2x\,dx \\ dv = e^x\,dx & v = e^x \end{bmatrix} = x^2e^x - 2\int xe^x\,dx = x^2e^x - 2xe^x + 2e^x + c,\ c \in \mathbb{R}$$

$$\mu = \int_0^1 \left(xe - xe^x\right)dx = \left(e\frac{x^2}{2} - e^x(x-1)\right)\Big]_0^1 = \frac{1}{2}e - 1$$

$$\sigma^2 = \int_0^1 \left(x^2e - x^2e^x\right)dx - \mu^2 = \left(e\frac{x^3}{3} - e^x\left(x^2 - 2x + 2\right)\right)\Big]_0^1 - \left(\frac{1}{2}e - 1\right)^2$$

$$= \left(\frac{e}{3} - e + 2\right) - \left(\frac{1}{2}e - 1\right)^2 = 2 - \frac{2e}{3} - \frac{e^2}{4} + e - 1 = 1 + \frac{e}{3} - \frac{e^2}{4}$$

d $\mathrm{P}(X \geqslant 0.5) = 1 - \mathrm{P}(X < 0.5) = 1 - \int_0^{\frac{1}{2}} \left(e - e^x\right)dx = 1 - \left(ex - e^x\right)\Big]_0^{\frac{1}{2}} = 1 - \left(\frac{1}{2}e - e^{\frac{1}{2}} + 1\right) = \sqrt{e} - \frac{e}{2}$

We can define a new variable: $Y \sim \mathrm{B}\left(n = 3, p = \sqrt{e} - \frac{e}{2}\right)$

i $\mathrm{P}(Y = 3) = \left(\sqrt{e} - \frac{e}{2}\right)^3$

ii If two batteries have failed, exactly one has not failed.

$\mathrm{P}(Y = 2) = 3\left(\sqrt{e} - \frac{e}{2}\right)^2\left(1 - \sqrt{e} + \frac{e}{2}\right)$

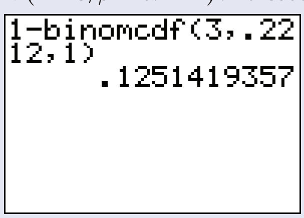
22 $f(y) = \begin{cases} 0 & y < 0 \\ 0.5e^{-\frac{y}{2}} & y \geqslant 0 \end{cases}$

a Since the time is measured in years, we need to calculate the following probability:

$\mathrm{P}(Y < 0.5) = \int_0^{0.5} 0.5e^{-\frac{y}{2}}dy = 0.5\left(-2e^{-\frac{y}{2}}\right)\Big]_0^{0.5} = -e^{-0.25} + 1 \approx 0.2212$

b We can use a binomial variable: $X \sim \mathrm{B}(n = 3, p = 0.2212)$. 'At least two' means either two or all three components fail.

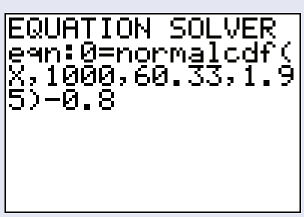
$\mathrm{P}(X \geqslant 2) = 1 - \mathrm{P}(X \leqslant 1) \approx 0.125$

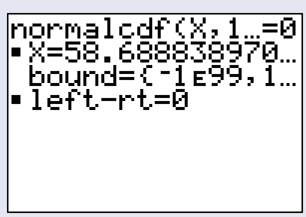


23 a $I \sim \mathrm{N}\left(\mu = 60.33, \sigma^2 = 1.95^2\right)$

$\mathrm{P}(I > x) = 0.8$

So, the distance is 58.69 m.

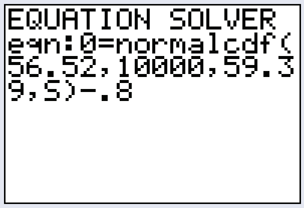


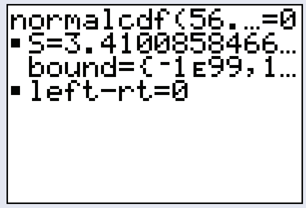


b $K \sim \mathrm{N}\left(\mu = 59.39, \sigma^2\right)$

$\mathrm{P}(K > 56.52) = 0.8$

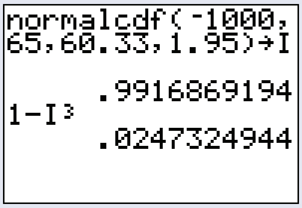
So, the standard deviation is 3.41 m.

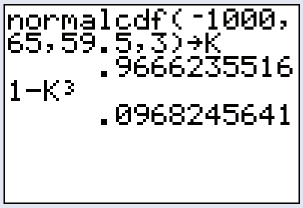




c $I \sim \mathrm{N}\left(\mu = 60.33, \sigma^2 = 1.95^2\right), K \sim \mathrm{N}\left(\mu = 59.50, \sigma^2 = 3^2\right)$

i We are going to use the complementary event, which is that neither of them has a throw longer than 65 m in those three throws.

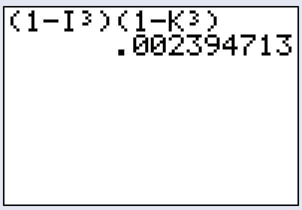




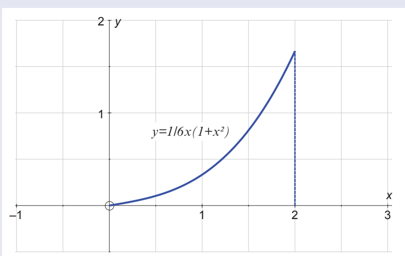
$1-(\mathrm{P}(I < 65))^3 = 0.0247$ $\quad$ $1-(\mathrm{P}(K < 65))^3 = 0.0968$

So, Karl is more likely to qualify.

ii



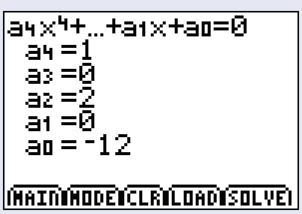
24 a

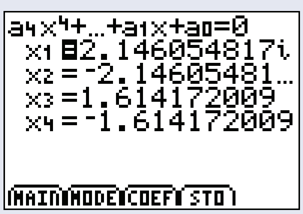


b From the graph, we can see that the modal value is 2 since the probability density function has a maximum value of $\frac{5}{3}$ at 2.

c $\int_0^2 \frac{1}{6}x^2\left(1+x^2\right)dx = \frac{1}{6}\left[\left(\frac{x^3}{3}+\frac{x^5}{5}\right)\right]_0^2 = \frac{1}{6}\left(\frac{8}{3}+\frac{32}{5}\right) = \frac{68}{45}$

d $\int_0^m \frac{1}{6}x\left(1+x^2\right)dx = \frac{1}{2} \Rightarrow \frac{1}{6}\left[\left(\frac{x^2}{2}+\frac{x^4}{4}\right)\right]_0^m = \frac{1}{2} \Rightarrow \frac{2m^2+m^4}{4} = 3 \Rightarrow m^4+2m^2-12=0$

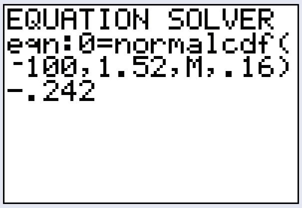


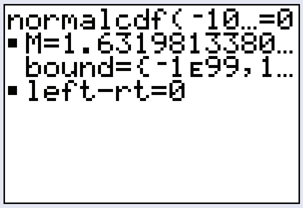


So, the median value is 1.614.

25 $A \sim \mathrm{N}\left(\mu = 1.56, \sigma^2 = 0.16^2\right), B \sim \mathrm{N}\left(\mu = 1.52, \sigma^2 = 0.16^2\right)$

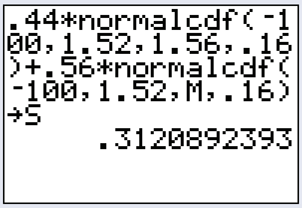
a $\mathrm{P}(B < 1.52) = 0.242$



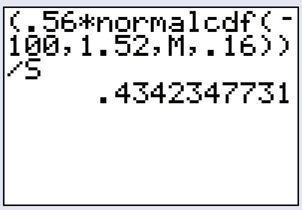


So, the mean diameter of a bolt produced by manufacturer B is 1.63 mm.

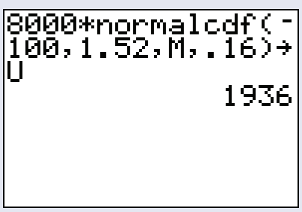
b $\mathrm{P}(S) = \underbrace{0.44}_{A} \cdot \mathrm{P}(A < 1.52) + \underbrace{0.56}_{B} \cdot \mathrm{P}(B < 1.52) = 0.312$

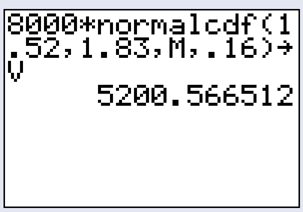


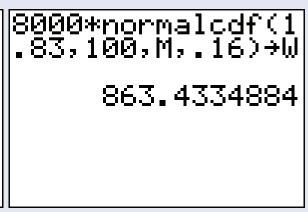
c $\mathrm{P}(B|S) = \dfrac{0.56 \cdot \mathrm{P}(B < 1.52)}{0.312} = 0.434$



d If manufacturer B makes 8000 bolts, the following are the expected numbers of bolts produced.







So, we can say that we would expect 1936 smaller bolts, 5201 bolts of diameter 1.52–1.83 mm and 863 larger bolts. Therefore, the expected profit is calculated as follows:

$5201 \times 1.5 + 863 \times 0.5 - 1936 \times 0.85 = 6587$ (correct to the nearest dollar).